Wave propagation through periodic structures in thin plates

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This thesis considers the propagation of flexural waves through structures embedded in thin, elastic plates. In a similar manner to photonic and phononic crystals, two-dimensional structures in thin elastic plates are known as platonic crystals (PlaCs) and can be formed using scatterers of any shape arranged in any regular periodic geometry. The time-harmonic waves that propagate through these structures are governed by the fourth-order biharmonic plate equation, in contrast to the second-order Helmholtz equation which governs wave propagation through other crystals and media, such as deep water. Here we consider two-dimensional structures which are comprised of circular scatterers, pins, and arbitrary shapes arranged in a square array. In addition to these two-dimensional PlaCs, we consider wave scattering by other platonic structures, such as one-dimensional arrays, finite clusters and single bodies. For circular and pinned geometries the solution can be obtained analytically using multipole methods, but for arbitrary shapes this is not possible. Here we outline a solution method for an arbitrarily shaped smooth scatterer using boundary integral equations, which arise from an appropriate decomposition of the biharmonic operator. The resulting system of equations can then solved by implementing boundary conditions at the body edge and using boundary element methods (BEMs). To verify the BEM solutions we use the multipole solutions for circular geometries, which are outlined here. For a two-dimensional array of arbitrarily shaped bodies one can construct the scattering and transfer matrices corresponding to a one-dimensional array, and then search for admissible Bloch factors. This permits the construction of band surfaces which reveal when Bloch-Floquet waves can propagate through the array to infinity. We also demonstrate that PlaCs can steer and disperse flexural waves analogously to light in photonic crystals and elastic waves in phononic crystals, and we validate this directly using localised Gaussian beams for pinned clusters. For such pinned structures the solution can be obtained directly using the fundamental solution to the biharmonic plate equation, and using these compact incident waves we are able to confirm the existence of a number of interesting diffraction effects, including negative refraction in thin plates. We also demonstrate that strong energy localisation is possible within the defects and waveguides of pinned PlaCs, and show a number of analogues to well known optical phenomena such as platonic polarisers in two-dimensional arrays of arbitrarily shaped scatterers. A solution is also presented for a single arbitrary body and for one-dimensional arrays of arbitrary geometry.
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Nomenclature

\( a \) Radius of a circular inclusion. .................................................. 32
\( a_l \) Coefficients in a Fourier expansion of the displacement ................. 38
\( A \) Area of the unit cell .............................................................. 177
\( A_b \) Area of the beam cross section ............................................ 8
\( \alpha_0 \) Horizontal component of the incident wave vector .................. 14
\( \alpha_m \) Horizontal component of the wave vector associated with the \( m \)th specular order. ................................................... 17
\( b_l \) Coefficients in a Fourier expansion of the normal derivative of the displacement ........................................ 38
\( B_l \) Bernoulli polynomial ......................................................... 173
\( C_m \) Fourier series coefficient for Hankel function expansion ............. 172
\( d \) Period of a one-dimensional array. .......................................... 13
\( \bar{d} \) Non-dimensionalised period of a one-dimensional array ............. 77
\( D \) Flexural rigidity of a plate ..................................................... 11
\( \delta(z) \) Dirac delta function. ..................................................... 34
\( \delta_0 \) Amplitude of a Helmholtz type incident wave ....................... 39
\( \hat{\delta}_0 \) Amplitude of a modified Helmholtz type incident wave ......... 39
\( \delta_{mn} \) Kronecker delta function. ............................................ 177
\( \Delta \) Two-dimensional Laplacian operator in Cartesian coordinates ....... 12
\( \bar{\Delta} \) Non-dimensionalised Laplacian operator. .......................... 77
\( E \) Young’s modulus ................................................................. 7
\( E_B \) Euler–Bernoulli ................................................................. 5
\( \eta \) An arbitrary parameter used for acceleration of array sums ............ 178
\( \epsilon_x \) Horizontal strain in a one-dimensional Euler–Bernoulli beam ....... 7
\( \epsilon_{pq} \) Strain tensor ............................................................... 11
\( f(x; t) \) Transverse loading of a beam or plate ................................ 8
\( F_l \) decomposed element of \( \mathcal{H}_l \) .......................................... 174
\( FEM \) Finite Element Method .................................................... 21
\( G \) Matrix which denotes the integral of the Green’s function over the panel \( \partial \Omega_q \) at the point \( \bar{x}_p \). ........................................... 25
\( GF \) Green’s function ............................................................... 17
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<tr>
<td>$\gamma$</td>
<td>Defined as $\gamma = \arg \xi$.</td>
</tr>
<tr>
<td>$\gamma_e$</td>
<td>Euler–Mascheroni constant, i.e. $\gamma = 0.577215$</td>
</tr>
<tr>
<td>$\gamma_m$</td>
<td>Defined as $\varphi_m = \arg r_m$.</td>
</tr>
<tr>
<td>$\gamma_{+,-,l,r}$</td>
<td>Edges of the fundamental cell.</td>
</tr>
<tr>
<td>$h$</td>
<td>Thickness of a Kirchoff–Love plate</td>
</tr>
<tr>
<td>$h_l$</td>
<td>Generalised form of the Schlomilch series $H_l$.</td>
</tr>
<tr>
<td>$H(x)$</td>
<td>Heaviside function.</td>
</tr>
<tr>
<td>$\mathbf{H}$</td>
<td>Matrix which denotes the integral of the normal derivative of the Green’s function over the panel $\partial \Omega_q$ at the point $x_p$.</td>
</tr>
<tr>
<td>$H_0^{(1)}$</td>
<td>Hankel function of the first kind, order zero.</td>
</tr>
<tr>
<td>$H_0^{(2)}$</td>
<td>Hankel function of the second kind, order zero.</td>
</tr>
<tr>
<td>$H_n^{(1)}$</td>
<td>Hankel function of the first kind, order $n$.</td>
</tr>
<tr>
<td>$\mathcal{H}_l$</td>
<td>Modified form of the grating sum $S - L^{H,G}$.</td>
</tr>
<tr>
<td>$I$</td>
<td>Second moment of inertia</td>
</tr>
<tr>
<td>$I_n$</td>
<td>Modified Bessel function of the first kind, order $n$.</td>
</tr>
<tr>
<td>$J_n$</td>
<td>Bessel function of the first kind, order $n$.</td>
</tr>
<tr>
<td>$\boldsymbol{\kappa}$</td>
<td>Bloch vector in two dimensions defined as $\boldsymbol{\kappa} = (\kappa_x, \kappa_y)$.</td>
</tr>
<tr>
<td>$\boldsymbol{\kappa}_s$</td>
<td>Arbitrary Bloch vector.</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Non-dimensional wave number</td>
</tr>
<tr>
<td>$k_j$</td>
<td>Basis vector for the reciprocal lattice vector $K_p$.</td>
</tr>
<tr>
<td>$K_p$</td>
<td>Reciprocal lattice vector</td>
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<tr>
<td>$K_0$</td>
<td>Modified Bessel function of the second kind, order $s$.</td>
</tr>
<tr>
<td>$K_n$</td>
<td>Modified Bessel function of the second kind, order $n$.</td>
</tr>
<tr>
<td>$K_L$</td>
<td>Kirchoff–Love</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of an Euler–Bernoulli beam</td>
</tr>
<tr>
<td>$M$</td>
<td>Moment of force for a one-dimensional beam.</td>
</tr>
<tr>
<td>$M_{pq}$</td>
<td>Moment tensor</td>
</tr>
<tr>
<td>$\Omega_b$</td>
<td>A small area in the cross section of a beam</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>The area between the scatterer boundary and the edges of the fundamental cell.</td>
</tr>
<tr>
<td>$\Omega_c$</td>
<td>The domain enclosed by our scatterer.</td>
</tr>
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The infinite domain surrounding our scatterer \( \Omega \) ........................................22
\( \rho \) Density of the material. ..................................8
\( \partial \Omega \) The boundary of our scatterer \( \Omega \) .........................22
\( \partial \Omega_q \) The \( q \)th panel of the discretised boundary \( \partial \Omega \) .................24
PDE Partial Differential Equation.............................................21
\( \phi \) Defined as \( \phi = \arg \mathbf{x}' \) ..................................35
\( \phi \) Vector which represents the discretised wave function \( w^H \) ..........24
\( \tilde{\phi} \) Vector which represents the discretised incident wave function \( w^H \) ........24
\( \gamma \) Vector which represents the discretised normal derivative of the wave function \( \partial_n w^H \) ..............................................25
\( \text{PlaC} \) Platonic Crystal. ..............................................1
\( r \) Norm of the field point vector \( \mathbf{x} \) .............................18
\( r' \) Norm of the vector \( \mathbf{x}' \) ........................................35
\( \mathbf{r}_j \) Basis vector of the array vector \( \mathbf{R}_p \) ..........................15
\( r_m \) Identical to \( \xi \) where \( \mathbf{x}' = (0, md) \) .................172
\( \mathbf{R}_p \) Array vector. ..................................................13
\( R(\mathbf{x}_p; t) \) Distance between the midpoint of the panel \( \partial \Omega_p \) and the parameterised source point. ...........................................26
\( S(\mathbf{x}_p; t) \) Function of field and source points. .......................26
\( S_{1,G}^H \) Grating sum of Helmholtz type (Scholmilch series). ..................35
\( S_{1,G}^K \) Grating sum of modified Helmholtz type (Scholmilch series) .........35
\( S_{1,A}^H \) Array sum of Helmholtz type (Scholmilch series). ..................43
\( S_{1,A}^K \) Array sum of modified Helmholtz type (Scholmilch series) ..........43
\( S_{1,A}^{V,A} \) Array sum of modified Helmholtz type (Scholmilch series) ..........43
\( \mathcal{S} \) Sum over all discrete integers minus a continuous integral. ..............173
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\( \sigma_x \) Horizontal stress in a one-dimensional Euler–Bernoulli beam .............7
\( \sigma_{pq} \) Stress tensor. ..................................................11
\( \tau_0 \) Imaginary part of \( \tilde{\lambda}_0 \) ....................................39
\( \theta \) Argument of the field point vector \( \mathbf{x} \) ............................18
\( \theta_i \) Incident wave angle. ..............................................14
\( \theta_m \) Angle of propagation associated with the \( m \)th specular order .........17
\( \vartheta_s \) Defined as \( \vartheta_s = \arg \kappa_s \) ....................................177
\( \Theta_h \) Defined as \( \Theta_h = \arg \mathbf{Q}_h \) ..................................17
\( \phi_p \) Defined as \( \phi_p = \arg \mathbf{R}_p \) ..................................43
\( u \) Displacement of the plate. .............................................6
\( v_q \) Length of the panel \( \partial \Omega_q \) ......................................25
\( u_x \) In-plane displacement in the \( x \) direction .............................10
\( u_x^0 \) In-plane displacement in the \( x \) direction of the plate midsurface ..........11
In-plane displacement in the $y$ direction. ........................................ 10
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Remark: this nomenclature applies to the content outside of the publications which partly constitute this thesis, and within these papers different notations may be present.
Introduction

“Platonic crystals could also be used to identify jewellery that you gift to a love interest who only likes you as a friend”

Alessandro Tuniz

The study of platonic crystals (PlaCs) is a relatively new field and is concerned with the propagation of elastic wave energy, in the form of bending waves, through thin elastic plates. It is primarily motivated by an extensive body of literature dedicated to the study of photonic crystals, which considers light propagation through periodic media of varying refractive index.

In this thesis we are concerned with the design of platonic structures, and in particular, whether particular PlaC designs can support wave energy propagation through the plate to infinity.

The PlaCs considered here are not formed by periodic variations in the material properties of the plate, instead they are fabricated using regularly spaced arrays of holes cut into the plate. At the edge of these cavities, we impose boundary conditions that model the different physical behaviours which can be imposed. These can range from fully clamped (zero displacement) to completely free-edge boundary conditions (that is, the plate can bend freely). In general, there is no restriction on the geometry of the cavity for our boundary integral solution outlined here, however for boundary element methods, we require only that the boundary be smooth. That said, there are certain simplifications for geometries such as circles where the solution can be obtained analytically.

An original motivation of this study was to understand wave propagation through large sheets of floating ice and very large floating structures (VLFS). There is considerable interest in the
behaviour of such structures, which are typically modelled as elastic plates which float with negligible submergence. These structures can possess numerous areas of open water and there is interest in how these inclusions affect the behaviour of the plate. This has several interesting applications, for example, it is possible that floating perforated plates could be used as artificial breakwaters to control destructive surface waves, and so may be useful in the design of future VLFS \cite{156}. In this thesis we are not concerned with the dynamics of fluid-loaded elastic plates \cite{103}, but instead with the problem of elastic wave propagation through plates which are surrounded above and below by a vacuum. The solution to this ‘dry problem’, or ‘in-vacuo’ problem, is of interest to sea-ice researchers, and contains sufficient complexity to warrant independent study. It is also of interest to engineers and physicists, particularly as a tool in vibration and wave energy control.

To avoid the added complexity of wave energy reflecting off the edges of a finite plate, we typically model our thin elastic plate as being of infinite extent and possessing at least one inclusion, or cavity. Obtaining the solution to the problem of wave scattering by a single cavity of arbitrary shape is quite challenging, and to the best of the author’s knowledge was demonstrated for the first time in Smith et al. \cite{135} for a plate in-vacuo. Existing work on in-vacuo plates was restricted to simple geometries such as circles \cite{67}, however Bennetts and Williams \cite{12} considered a single arbitrary cavity in a floating elastic plate. Additional complexity arises when multiple bodies are considered, such as one-dimensional or two-dimensional arrays, and solutions can be found by applying and extending several concepts from optics, including diffraction grating theory and Bloch–Floquet theory.

We are especially interested in the design of platonic diffraction gratings and arrays, which can be used to steer and disperse flexural waves. Here a diffraction grating is defined as a regular periodic structure of infinite extent (i.e., a one-dimensional array) which can be used to split and diffract incident plane waves into multiple plane waves \cite{13}. Each of these refracted plane waves then propagate in different directions at different wavelengths (i.e., speeds). The relationship between wavelength, grating period, and specular order is determined using the Fraunhofer grating equation, which is used for propagating incident waves that satisfy the Helmholtz equation. A modified version of this is required for modified Helmholtz incident waves. The dispersive properties of diffraction gratings make them very useful tools with wide applications from wave filtering to demultiplexing and spectrometry \cite{13}. An understanding of diffraction gratings is relevant to the study of wave propagation through two-dimensional lattices, as any doubly periodic array can be regarded as an infinite stack of gratings. Alternatively, we can achieve the solution to a two-dimensional array problem directly by the use of Bloch–Floquet theory for particular geometries. This approach has some advantages as we can obtain a dispersion relation for the PlaC straightforwardly. Using either approach we can obtain the spectral band surfaces of the PlaC, which reveal when wave propagation is supported through a two-dimensional lattice. From a practical perspective, researchers can only create large finite arrays and attempt to couple into the Bloch modes associated with theoretically infinite structures. Evaluating the solution directly for such finite structures, commonly known as platonic clusters, can be difficult.
when the scatterer is of arbitrary geometry, as we cannot rely on certain conditions to reduce the complexity of the problem. However for certain geometries such as pins, the solution can be obtained directly. We find that the predicted scattering behaviours of PlaCs can be observed for large platonic clusters of pinned points, using compact incident waves.

The term *platonic crystal* was first coined in McPhedran et al. [99] where it was defined as any structured system that is governed by the biharmonic plate equation. The term ‘platonic’ is derived directly from the word plate (i.e., a shell of zero curvature) and was chosen to be phrased analogously to photonic, phononic and plasmonic crystals. The earliest references found to periodic structures embedded in thin plates are the works by Kouzov and Lukyanov [69, 70] who consider a doubly periodic array of pins and point masses for both plates in-vacuo and surrounded by an acoustic medium on one side. The solution to the pinned plate problem was derived independently some time later by Evans and Meylan [35] and Evans and Porter [38], and these are widely regarded as pivotal works in the literature. PlaCs comprising of pins are of considerable interest to researchers due to the elegance and simplicity of the solution procedure. This arises from the linearity of the biharmonic plate equation which permits the scattered field to be expressed directly in terms of the Green’s function for the plate. Recent work has demonstrated their abilities in wave trapping, electromagnetically induced transparency effects, and their ability to act analogously to lenses (i.e., as a flexural lens) which gives rise to interesting optical-type behaviours for flexural waves in plates [53, 54, 136]. Attention has also been paid to the resonances of pinned platonic clusters (finite PlaCs) in Meylan and McPhedran [104] where the solution in the time domain is presented for various incident wave types using the singularity expansion method. Note that in a practical context these pins can be well approximated by using suitably large point mass loadings, as discussed in Evans and Porter [38], and that there is an extensive body of research on arrays of simply-supported plates and beams dating back to the early 1970’s [100, 102, 130].

Other recent work on PlaCs has focused on engineering structures which exhibit complex diffraction behaviours. This includes the seminal work by Farhat et al. [42] who examined a doubly periodic array of square inclusions and were able to theoretically demonstrate the existence of negative refraction in thin plates. This was demonstrated using a finite cluster of square inclusions to successfully refocus a point source on one side of the structure. The ability of PlaCs to successfully induce negative refraction was also demonstrated in Smith et al. [136] for a pinned array of points using Gaussian beams. The first experimental work on platonic cloaking was done by Stenger et al. [145] who, influenced by Farhat et al. [42], constructed a concentric ring cloak design around a single clamped circular scatterer. They were able to successfully demonstrate cloaking over a wide frequency interval (200 to 400 Hz) by appropriate variations in the Young’s modulus throughout the cloak design. There have also been novel platonic structure designs, such as in Colquitt et al. [26, 27], which has investigated regular two-dimensional arrays of masses that are connected with Euler–Bernoulli beams as the lattice links. These structures have been shown to exhibit anisotropy at high frequencies and strong energy localisation.
Also of interest are platonic structures comprised of circular inclusions. The earliest known solution to the problem of a single circular inclusion was given in Konenkov [67] using multipole techniques (see also Leissa [73], Norris and Vemula [111], and Andronov [3] using zero range potentials). Similarly, the first known work investigating the plate response to a single grating of circular inclusions was presented in Movchan et al. [107], subject to clamped and free-edge boundary conditions. In this work they also examined the case of vanishing radius (i.e., pins) as well as multiple grating stacks. Likewise, in Movchan et al. [106] square arrays of circular scatterers were considered, subject to clamped and free-edge boundary conditions. They also considered the case of vanishing radius and constructed several band diagrams for both problems. This publication was then followed by Poulton et al. [124] who performed an exhaustive investigation into circular scatterers of different radius (for a square array of period unity) and gave highly converged band surface diagrams in addition to useful tables of values. Non-regular PlaCs, such as those constructed as a random array of circular inclusions, have been examined in Parnell and Martin [115]. Other non-regular PlaCs, such as defective PlaCs, have been considered by Poulton et al. [125] for pins and point masses, and also by Smith et al. [137] for the case of pins alone. Other complex platonic designs, such as checkerboard structures and periodic microstructured plates have been investigated [6, 41], as have structured plates which are governed by Mindlin plate theory as opposed to Kirchoff–Love plates [108]. For the two-dimensional array problems considered in this thesis, we restrict our attention to square and rectangular array designs. The extension to other array geometries can be made easily using the general expression for the quasiperiodic Green’s function which can be found in McPhedran et al. [98].

This thesis is primarily comprised of four central publications which cover the design of various PlaCs. The first paper deals with the construction of platonic clusters which are fabricated using zero-radius circular scatterers, or pins. The paper outlines how to construct the band surfaces for a doubly periodic array of pins, and then examines the particular case of a square array, investigating the diffraction behaviour of a finite square cluster using Gaussian beams. We are able to demonstrate that these finite clusters can focus flexural wave energy to achieve several interesting diffraction behaviours including negative refraction. The second paper is concerned with the construction of defective pinned PlaCs. In particular, we investigate wave scattering and wave trapping by defective platonic gratings and arrays when we remove several configurations of pins from an otherwise perfect grating or doubly periodic rectangular array. These include the removal of a single pin, a square cluster of pins, and entire lines of pins (otherwise known as waveguides) for the rectangular array problem, and the removal of multiple points from the one-dimensional array problem. The third paper considers the problem of wave scattering by a single body of arbitrary geometry embedded in a thin plate. We provide a solution outline for the cases when free-edge, simply-supported and clamped-edge boundary conditions are imposed at the boundary of the scatterer. The solution is obtained using boundary integral techniques, and solved numerically using boundary element methods. We also provide an interesting connection to the problem of plate vibration, and compute the modes associated with circular and elliptical plate geometries. The fourth paper deals with one- and two-dimensional
square arrays comprised of scatterers which are of arbitrary shape, but we consider clamped-
edge boundary conditions alone. We examine a number of scatterer geometries, whose first
band surfaces exhibit varying degrees of curvature, and demonstrate the existence of platonic
polarisers as well as other interesting diffraction phenomena.

We begin the thesis by providing an introduction to plate and optical theory, as well as outlining
the derivation of some important Green’s functions. An outline of boundary element theory,
which is used to solve several systems of boundary integral equations within this thesis, is also
provided. In Chapter 2 we provide multipole solutions to various problems involving circular
scatterers — in particular, wave scattering by a single body, grating, and wave propagation
through a two-dimensional square array. Chapter 3 is concerned with the construction of pinned
PlaCs which is based on the first chapter, Chapter 4 examines defective PlaCs as per the second
paper, and Chapter 5 deals with arbitrarily shaped scatterers which is framed using the third
and fourth papers. Finally we provide concluding remarks in Chapter 6 and provide an outline
of future research directions. Derivations of grating and array sums are given in the appendices.

1.1 Concepts from the theory of plates and shells

This section provides an outline for the derivation of the beam and plate equations that are
used in this thesis. In particular, we consider the derivation of the governing partial differential
equations which model thin, elastic beams and plates. In two dimensions this is known as
the Kirchoff–Love (KL) plate equation and in one dimension, the Euler–Bernoulli (EB) beam
equation.

1.1.1 Derivation of the Euler–Bernoulli beam equation

From any undergraduate textbook in engineering or physics one can easily understand the
concept of a force as being a vector quantity which moves a body from rest or existing uniform
motion. It is also straightforward to understand that any force applied to a body causes a
rotation of that body about a point — an effect known as a moment of a force (which is also
a vector quantity). Using the ideas of forces and moments we can investigate the response of
bodies to stress (force per unit area), which can take the form of rigid body displacement (simple
translations and rotations) as well as deformations (changes in the size and shape of the body).

We are interested in considering the response of an isotropic thin plate (i.e., with uniform
material properties in all directions) and begin with the derivation of the EB beam equation;
which is derived using a simplified version of linear elasticity theory and considers lateral loadings
only. In particular, the model assumes that the vertical displacement of the plate is small relative
to the thickness, that the thickness is unchanged during deformation, and that straight lines
normal to the mid-plane of the beam remain straight, normal, and of identical magnitude after deformation [45], as seen in Figure 1.1.

During bending, we have the presence of normal and transverse (non-normal) stresses, and we refer to stress parallel to the cross section of the beam as shear stress. Essentially for our loaded beam we assume a stretching action which gives rise to a horizontal displacement and a bending contribution which gives rise to vertical displacement/translation [33]. In the EB framework we assume that the horizontal (in-plane) displacement is negligible.

The procedure for deriving the EB and KL equations are given in Doyle [32], Dym and Shames [33], Goodwine [45], Wang et al. [155] and we provide a brief outline here. The technique begins by considering the kinematic condition which gives the relationship between the strain and the vertical displacement of the beam centre line, $w_0$.

Note that in the literature, reference is often made to deflection (which in this context represents displacement of the beam under loading), strain (which is essentially a normalised measure of deformation) and shear (which typically refers to shear strain — the deformation of the body in response to shear stress). It is important to distinguish these parameters.

### 1.1.2 The kinematic condition

In reference to Figure 1.2, we can examine a small segment of the beam from $x$ to $x + \Delta x$ and consider the behaviour near an arbitrary vertical distance $z$ from the centre line.

The horizontal displacement $u$, (i.e., how far the cross section has been extended relative to the rest state) can then be written as

$$\sin(\phi) = \frac{u}{z}, \quad (1.1.1)$$

where $\phi$ is the angle at the edge of the beam as shown in Figure 1.2.

Over an arbitrarily small section of beam (from $x$ to $x + \Delta x$) we have a total horizontal dis-
placement at elevation $z$ of

\[ u = z \sin [\varphi(x + \Delta x)] - z \sin [\varphi(x)], \]  

(1.1.2)

and since strain is defined as displacement per unit length one can write

\[ \epsilon_x = \frac{z \sin [\varphi(x + \Delta x)] - z \sin [\varphi(x)]}{\Delta x}. \]  

(1.1.3)

Assuming the angle is small we have $\sin(\varphi) \simeq \varphi$, and from the Kirchoff assumption for rotations $\varphi$ about the mid surface, we have $\varphi \simeq \partial_x u$ revealing

\[ \epsilon_x = z \left[ \partial_x u(x + \Delta x) - \partial_x u(x) \right]. \]  

(1.1.4)

If we take a first order Taylor series expansion about the point $x$ one can obtain

\[ \partial_x u(x + \Delta x) = \partial_x u(x) + \Delta x \partial_x^2 u(x) + O(\Delta x^2), \]  

(1.1.5)

and subsequently (1.1.4) can be expressed in the form

\[ \epsilon_x = z \partial_x^2 u(x). \]  

(1.1.6)

1.1.3 The constitutive equation

The constitutive condition gives the relationship between stress and strain, which we assume here to be appropriately modelled by Hooke’s law [45]

\[ \sigma_x = E \epsilon_x, \]  

(1.1.7)

where $E$ denotes the Young’s modulus of the material. Substituting this into (1.1.6) reveals the strain-displacement relation

\[ \sigma_x = E z \partial_x^2 u. \]  

(1.1.8)
1.1 Concepts from the theory of plates and shells

1.1.4 The equilibrium equations

Here we consider a vertical cross section of a beam at an arbitrary height $z$ from the midsection, and examine a small area about that point which we denote by $\Omega_b$. The dominant stress $\sigma_x$ gives rise to a moment over this small area, which can be represented as the integral

$$M = - \int_{\Omega_b} z \sigma_x \, dxdz. \quad (1.1.9)$$

Integrating the stress alone over $\Omega_b$ admits the shear resultant

$$V = - \int_{\Omega_b} \sigma_x \, dxdz, \quad (1.1.10)$$

which denotes the transverse shear over the cross section.

If we examine a small horizontal cross section of the beam, then from Newton’s first law one can obtain

$$\rho A_b \frac{\partial^2 u}{\partial t^2}(x + \Delta x; t) \Delta x = V(x + \Delta x; t) - V(x; t) + \frac{\Delta x}{2} \left( f(x + \Delta x; t) + f(x; t) \right). \quad (1.1.11)$$

Here the load is taken to be the average over the infinitesimal section, $A_b$ denotes the area of the cross section and $\rho$ denotes the density of the material.

If we take a first order Taylor series expansion of the functions $\partial_t u(x; t)$, $V(x; t)$ and $f(x; t)$ about the point $x$, one can obtain

$$\partial_t^2 u(x + \Delta x; t) = \partial_t^2 u(x; t) + \Delta x \partial_x \partial_t^2 u(x; t) + O(\Delta x^2), \quad (1.1.12a)$$

$$V(x + \Delta x; t) = V(x; t) + \Delta x \partial_x V(x; t) + O(\Delta x^2), \quad (1.1.12b)$$

$$f(x + \Delta x; t) = f(x; t) + \Delta x \partial_x f(x; t) + O(\Delta x^2). \quad (1.1.12c)$$

If we then take the limit as $\Delta x \to 0$, then (1.1.11) becomes the first equilibrium equation

$$\rho A_b \partial_t^2 u = \partial_x V + f. \quad (1.1.12d)$$

In reference to Figure 1.3 one can balance forces at $x + \Delta x$ to obtain the expression

$$M(x + \Delta x; t) - M(x; t) - V(x; t) \Delta x + \frac{\Delta x^2}{4} \left( f(x; t) + f(x + \Delta x; t) \right) = 0, \quad (1.1.13)$$

where the moment generated from the loading is taken to be the average load multiplied by the average moment arm $\Delta x/2$ [45].

If we take a first order Taylor series expansions of the functions $M(x; t)$ and $f(x; t)$ about the
1.1 Concepts from the theory of plates and shells

\( f(x; t) \)

\( M(x; t) \)

\( (x + \Delta x; t) \)

\( V(x; t) \)

\( M(x; t) \)

\( V(x + \Delta x; t) \)

\( x \)

\( x + \Delta x \)

Figure 1.3: A segment of an EB beam outlining shear resultants \( V(x; t) \), forces \( f(x; t) \) and moments \( M(x; t) \).

point \( x \), one can obtain

\[
M(x + \Delta x; t) = M(x; t) + \Delta x \partial_x M(x; t) + O(\Delta x^2), \tag{1.1.14a}
\]

\[
f(x + \Delta x; t) = f(x; t) + \Delta x \partial_x f(x; t) + O(\Delta x^2). \tag{1.1.14b}
\]

If we then take the limit as \( \Delta x \to 0 \), (1.1.13) becomes the second equilibrium equation

\[
\partial_x M = V. \tag{1.1.15}
\]

By differentiating (1.1.15) with respect to \( x \) and then substituting the result into (1.1.13) one obtains

\[
\rho A_b \partial_t^2 u = \partial_x^2 M + f(x; t). \tag{1.1.16}
\]

We can then substitute (1.1.8) into (1.1.9) revealing

\[
M = - \int_{\Omega_b} z \sigma_x dydz = -E \partial_x^2 u \int_{\Omega_b} z^2 dydz = -EI \partial_x^2 u, \tag{1.1.17}
\]

where

\[
I = \int_{\Omega_b} z^2 dydz, \tag{1.1.18}
\]

and \( I \) is defined as the second moment of inertia. This admits the EB beam equation under lateral loading \( f(x; t) \):

\[
\rho A_b \partial_t^2 u + \partial_x^2 (EI \partial_x^2 u) = f(x; t). \tag{1.1.19}
\]

For the free vibration case (i.e. zero loading, or \( f(x; t) = 0 \)) and assuming the beam is isotropic (i.e., \( E \) and \( I \) are constants) this simplifies to

\[
\partial_x^4 u + \alpha^2 \partial_t^2 u = 0, \tag{1.1.20}
\]

where \( \alpha = \sqrt{\rho A_b / EI} \). This can be further simplified if we consider time-harmonic solutions of
1.1 Concepts from the theory of plates and shells

the form \( u(x) = \text{Re} \left\{ w(x)e^{ikx} \right\} \) which admits

\[
\frac{\partial^4 w}{\partial x^4} - k^4 w = 0,
\]

(1.1.21)

where \( k = \sqrt{\alpha \omega} \).

1.1.5 Boundary conditions for an Euler–Bernoulli beam

Having derived the fourth-order partial differential equation which governs the dynamics of our EB beam, it is straightforward to show that the homogeneous solution takes the form

\[
w(x) = A \sin(kx) + B \cos(kx) + C \sinh(kx) + D \cosh(kx),
\]

(1.1.22)

where the amplitudes \( A, B, C \) and \( D \) are unknown. In order to determine the specific solution it is therefore necessary to impose four boundary conditions. These conditions are usually applied at the ends of the beam (taken to be of length \( L \)) and can include clamped-edge, free-edge or simply-supported edge conditions. For example, we could specify clamped-edge conditions at both ends:

\[
\begin{align*}
w\big|_{x=0} &= 0, & \partial_x w \big|_{x=0} &= 0, \\
w\big|_{x=L} &= 0, & \partial_x w \big|_{x=L} &= 0,
\end{align*}
\]

(1.1.23a, 1.1.23b)

free-edge conditions at both ends:

\[
\begin{align*}
\partial_x^2 w \big|_{x=0} &= 0, & \partial_x^2 w \big|_{x=0} &= 0, \\
\partial_x^2 w \big|_{x=L} &= 0, & \partial_x^3 w \big|_{x=L} &= 0,
\end{align*}
\]

(1.1.24a, 1.1.24b)

or simply-supported conditions at both ends:

\[
\begin{align*}
w\big|_{x=0} &= 0, & \partial_x^2 w \big|_{x=0} &= 0, \\
w\big|_{x=L} &= 0, & \partial_x^2 w \big|_{x=L} &= 0,
\end{align*}
\]

(1.1.25a, 1.1.25b)

[45]. Other more complicated boundary conditions may also be specified, and it is also possible to consider mixed boundary conditions.

1.1.6 Derivation of the Kirchoff–Love thin plate equation

We now extend the theory to a KL plate in flexure. Note that in some publications, the EB equation is used to refer to both one- and two-dimensional thin, elastic plates.

This derivation begins by introducing the displacement vector \((u_x, u_y, u_z)\) where \(u_x\) and \(u_y\) are
1.1 Concepts from the theory of plates and shells

in-plane displacements and $u_z$ represents the out-of-plane displacement. Analogous to the beam case, we expand these as follows

$$u_x = u_x^0 - z\partial_x u, \quad (1.1.26a)$$
$$u_y = u_y^0 - z\partial_y u, \quad (1.1.26b)$$
$$u_z = u, \quad (1.1.26c)$$

and we assume that the in-plane displacements of the centre line are negligible ($u_x^0 = u_y^0 = 0$).

Assuming the plate is isotropic then we have the stress-strain (constitutive) condition [33]

$$
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix} \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\epsilon_{xy}
\end{bmatrix},
$$

(1.1.27)

which is a generalised form of Hooke’s law in two dimensions. Note that $\sigma_{xy} = \sigma_{yx}$ and $\epsilon_{xy} = \epsilon_{yx}$ and so only three parameters are considered here.

The strain-displacement (kinematic) condition is given by

$$
\begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\epsilon_{xy}
\end{bmatrix} = \begin{bmatrix}
z\partial_x^2 u \\
z\partial_y^2 u \\
z\partial_{xy}^2 u
\end{bmatrix},
$$

(1.1.28)

[33] and as before we can define the moments

$$
\begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix} = -\int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} z dz = -\frac{2h^3 E}{3(1-\nu^2)} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix} \begin{bmatrix}
z\partial_x^2 u \\
z\partial_y^2 u \\
z\partial_{xy}^2 u
\end{bmatrix},
$$

(1.1.29)

where $h$ denotes the thickness of the plate.

From Newton’s first law we can obtain the equilibrium equation

$$\partial_x^2 M_{xx} + 2\partial_{xy}^2 M_{xy} + \partial_y^2 M_{yy} - 2\rho h \partial_t^2 u = f(x, t),
$$

(1.1.30)

where forces associated with the transverse acceleration of the plate are taken to be negligible. Differentiating the system (1.1.29) one can obtain

$$
\begin{align*}
\partial_x^2 M_{xx} &= -D (\partial_x^4 u + \nu \partial_x^2 \partial_y^2 u), \\
\partial_y^2 M_{yy} &= -D (\nu \partial_x^2 \partial_y^2 u + \partial_y^4 u), \\
\partial_{xy}^2 M_{xy} &= -D (1-\nu) \partial_x^2 \partial_y^2 u,
\end{align*}
$$

(1.1.31)

where $D = 2h^3 E / 3(1-\nu^2)$. Substituting (1.1.31a) to (1.1.31c) into (1.1.30) reveals the KL plate
equation directly, which is given by

\[(D\Delta^2 + \rho h \partial_t^2) u = f(x; t),\]  

(1.1.32)

where \(\Delta = \partial_x^2 + \partial_y^2\) is the two-dimensional Laplace operator in Cartesian coordinates. Typically we are interested in time-harmonic solutions \(u(x) = \text{Re}\{w(x)e^{i\omega t}\}\) which admits the frequency domain representation

\[(\Delta^2 - k^4) w = 0,\]  

(1.1.33)

for the case of zero forcing (i.e., the in-vacuo problem), where \(k^4 = \omega^2 \rho h / D\). Note that functions which satisfy \(\Delta^2 w = 0\) are known as biharmonic functions, and as such the KL equation (1.1.33) is often referred to as the biharmonic plate equation.

1.1.7 Boundary conditions for a Kirchoff–Love plate

The solution to the KL equation depends strongly on the geometry of the plate and here we provide an outline of the different boundary conditions that can be imposed on a finite circular plate as demonstrated in Leissa [73]. These boundary conditions are given in polar coordinates, and we begin by by defining the bending and twisting moments

\[
M_r = -D \left( \partial_r^2 w + \nu \left[ \frac{1}{r} \partial_r w + \frac{1}{r^2} \partial_r^2 w \right] \right),
\]

(1.1.34a)

\[
M_\theta = -D \left( \frac{1}{r} \partial_r w + \frac{1}{r^2} \partial_\theta^2 w + \nu \partial_r^2 w \right),
\]

(1.1.34b)

\[
M_r \theta = -D(1 - \nu) \partial_r \left( \frac{1}{r} \partial_\theta w \right),
\]

(1.1.34c)

as well as the transverse shearing forces

\[
V_r = -D \partial_r (\Delta w),
\]

(1.1.35a)

\[
V_\theta = -D \frac{1}{r} \partial_\theta (\Delta w).
\]

(1.1.35b)

Accordingly, for a circular plate of radius \(a\) one can impose the condition that the plate is clamped all around the edge:

\[w \big|_{r=a} = 0, \quad \partial_r w \big|_{r=a} = 0,\]  

(1.1.36a)

or one can impose free-edge edge conditions:

\[M_r \big|_{r=a} = 0, \quad V_r \big|_{r=a} + \frac{1}{r} \partial_\theta M_r \theta \big|_{r=a} = 0,\]  

(1.1.36b)
or even impose simply-supported boundary conditions:

\[ w|_{r=a} = 0, \quad M_r|_{r=a} = 0. \quad (1.1.36c) \]

Multipole solutions to problems involving circular inclusions are discussed in depth in Chapter 2.

Having derived the necessary PDEs which govern our platonic structures in one and two dimensions, we now turn our attention to certain applicable principles of optics.

### 1.2 Concepts from Optical Theory

#### 1.2.1 Quasiperiodic functions and conditions

The term quasiperiodic arises from the definition of a quasiperiodic function – i.e., a function which is similar in nature to a periodic function, \( f(x + d) = f(x) \), except that across each period we have a change in phase \( f(x + d) = f(x)e^{i\kappa x d} \) where \( \kappa x \) is known as the Bloch factor. Such functions arise often in problems involving regular arrays and permit a reduction in the domain down to a single period (for a one-dimensional array), or a fundamental cell (for a two-dimensional array).

For a non-defective two-dimensional array, the fundamental cell about the origin is known as the primitive, or Wigner–Seitz cell. To determine the displacement at any point in the array relative to the central cell, we can use the quasiperiodicity condition

\[ w(x + R_p) = w(x)e^{i\kappa \cdot R_p}, \quad (1.2.1) \]

where \( x = (x, y) \), \( R_p \) is the array vector which gives the central coordinate location of each scatterer in the array (i.e., for a square array \( R_p = (md, nd) \) where \( m, n \in \mathbb{Z} \) and \( d \) is the period) and \( \kappa \) denotes an arbitrary Bloch vector. This condition essentially states that there is only a phase difference in the solution in the cell centred at the point \( R_p \), relative to the cell centred about the origin. Solutions which satisfy this condition are known as Bloch waves, or Bloch–Floquet waves. From this condition one can then construct a dispersion relation expression which reveals the band surfaces of a structure. These band surfaces are a very useful visual tool for determining not only when wave propagation is supported through the array, but also the possible directions of wave energy propagation through the crystal.

At particular frequencies and angles of incidence, it is possible for a plane wave that strikes the edge of an array to excite Bloch waves inside the array. This is achieved by a suitable coupling of the incident plane wave with the Bloch modes of the structure. In cases where this is not possible, all of the wave energy is reflected and we say that there exists a partial band gap at
1.2 Concepts from Optical Theory

that given incident angle and frequency. A complete, or full, band gap is said to exist when wave energy cannot penetrate through the array for all angles of incidence at a given frequency.

Likewise for one-dimensional arrays we can impose the quasiperiodicity condition

\[ w(x + md) = w(x) e^{i\alpha_0 md}, \quad \text{where } m \in \mathbb{Z}, \]  

(1.2.2)

which corresponds to an array situated along the \( x \)-axis (with period \( d \)) subject to a plane incident wave at angle \( \theta_i \) and wave number \( k \), where \( \alpha_0 = k \sin \theta_i \).

1.2.2 Reciprocal Lattices and the Brillouin Zone

In this section we provide an outline of how to relate doubly periodic array problems to their associated Fourier representations. It is well known that when the Fourier transform of a suitable function is taken, the function is no longer represented in terms of real space but instead in terms of reciprocal (or Fourier) space. Accordingly, for any regular lattice in real space there exists a corresponding representation in reciprocal space. This representation is known as the reciprocal lattice.

Additionally, when a quasiperiodicity condition is used on any lattice geometry, it allows us to reduce our attention down to a fundamental, or Wigner–Seitz, cell in real space. The associated fundamental cell in reciprocal space is known as the (first) Brillouin zone, and we outline how to geometrically determine this here. It is also possible to construct higher order Brillouin zones. However these are not typically used in diffraction problems, but are useful in solid state physics [63]. These Brillouin zones reveal frequencies where incident waves can be Bragg-reflected by an array. In Figure 1.4 we present an image of a truncated square array centred around the origin in both Cartesian and reciprocal space.

Further details on the content from this section can be found in Joannopoulos [57] and we provide a brief summary here, which begins by considering a periodic function in three dimensions, i.e., a function \( f(\mathbf{x}) \) which satisfies the condition

\[ f(\mathbf{x} + \mathbf{R}_p) = f(\mathbf{x}), \]  

(1.2.3)

where \( \mathbf{x} = (x, y, z) \) and \( \mathbf{R}_p \) is the array vector. The function \( f \) has the associated Fourier transform

\[ f(\mathbf{x}) = \int_{-\infty}^{\infty} g(\mathbf{\kappa}) e^{i\mathbf{\kappa} \cdot \mathbf{x}} d\mathbf{\kappa}, \]  

(1.2.4)

where \( \mathbf{\kappa} = (\kappa_x, \kappa_y) \). If we take the Fourier transform of the function some periods away we obtain

\[ f(\mathbf{x} + \mathbf{R}_p) = \int_{-\infty}^{\infty} g(\mathbf{\kappa}) e^{i\mathbf{\kappa} \cdot \mathbf{R}_p} e^{i\mathbf{\kappa} \cdot \mathbf{x}} d\mathbf{\kappa}, \]  

(1.2.5)
Figure 1.4: Outline of a two-dimensional square array in (a) Cartesian and (b) reciprocal space, with the Brillouin zone and irreducible Brillouin zones shaded.

Equating the two expressions (1.2.5) and (1.2.4) one can then obtain

\[ g(\kappa) = g(\kappa) e^{i\kappa \cdot R_p}, \]  

which can only be satisfied provided that \( g(\kappa) = 0 \) or \( e^{i\kappa \cdot R_p} = 1 \). We can satisfy the second condition providing \( \kappa \cdot R_p = 2\pi n, \ n \in \mathbb{Z} \). The set of vectors \( \kappa \) which satisfy this condition are known as reciprocal lattice vectors, and form their own lattice in reciprocal space. We refer to the lattice in \((x, y, z)\) space as the primitive lattice, with an associated primitive lattice vector \( R_p \). For the case of a rectangular lattice in three dimensions, we have

\[ R_p = i \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + j \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix} + k \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix} = i r_1 + j r_2 + k r_3, \]  

where \( i, j, k \in \mathbb{Z} \) and \( p \) represents multi-index notation. If we represent the reciprocal lattice vector (unknown at present) in the form

\[ K_q = r k_1 + s k_2 + t k_3, \]  

where \( q = (r, s, t) \in \mathbb{Z}^3 \), then we can represent our condition \( \kappa \cdot R_p = 2\pi n \) as \( K_q \cdot R_p = 2\pi n \), or

\[ (r k_1 + s k_2 + t k_3) \cdot (i r_1 + j r_2 + k r_3) = 2\pi n. \]  

This condition is satisfied provided \( k_i \cdot r_j = 2\pi \delta_{ij} \). Using the scalar triple product vector identity

\[ r_1 \cdot (r_2 \times r_3) = r_2 \cdot (r_3 \times r_1) = r_3 \cdot (r_1 \times r_2), \]  

(1.2.10)
we construct the following ansatz

\[ k_1 = A(r_2 \times r_3), \quad k_2 = B(r_3 \times r_1), \quad k_3 = C(r_1 \times r_2). \]  

(1.2.11)

Using the condition \( r_i \cdot k_j = 2\pi \delta_{ij} \) and (1.2.10) one can easily determine the \( A, B \) and \( C \) coefficients to reveal

\[ k_1 = \frac{2\pi (r_2 \times r_3)}{r_1 \cdot (r_2 \times r_3)}, \quad k_2 = \frac{2\pi (r_3 \times r_1)}{r_1 \cdot (r_2 \times r_3)}, \quad k_3 = \frac{2\pi (r_1 \times r_2)}{r_1 \cdot (r_2 \times r_3)}. \]  

(1.2.12)

One advantage of considering problems in reciprocal space is that we have translational symmetry. That is, the vectors \( \kappa \) and \( \kappa + K \) correspond to the same Bloch mode.

For a square array in two dimensions we can use the same formulae (1.2.12) to determine the reciprocal lattice vectors by specifying

\[
\begin{aligned}
\mathbf{r}_1 &= \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}, & \mathbf{r}_2 &= \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix}, & \mathbf{r}_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
\end{aligned}
\]  

(1.2.13)

to obtain

\[
\begin{aligned}
k_1 &= \frac{2\pi}{d} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & k_2 &= \frac{2\pi}{d} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & k_3 &= \frac{2\pi}{d} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}
\]  

(1.2.14)

That is, for a square array in two dimensions we have

\[
\begin{aligned}
R_p &= d(m, n), \quad \text{where} \quad (m, n) \in \mathbb{Z}_2, \quad \text{and} \\
K_q &= \frac{2\pi}{d} (r, s), \quad \text{where} \quad (r, s) \in \mathbb{Z}_2,
\end{aligned}
\]  

(1.2.15, 1.2.16)

as the array vectors in real and Fourier space. Note that for any two-dimensional lattice we can specify \( r_3 = (0, 0, 1) \) to obtain the associated reciprocal lattice vector.

An understanding of reciprocal space, and the different lattice vector representations that arise, is especially important in understanding the complicated Green’s function expressions which arise in this thesis.

Alternatively, one can construct the geometry of the Brillouin zone [19] by drawing the perpendicular bisecting lines corresponding to the first few reciprocal lattice vectors. The enclosed area is known as the first Brillouin zone. Inside this zone we can use symmetry to deduce the irreducible Brillouin zone which is the smallest area required to construct the band surfaces of a crystal over the entire first Brillouin zone. For the case of a square array, this is given by the triangular wedge seen in Figure 1.4(b). The picture over the whole Brillouin zone is then obtained by repeated reflection of this wedge.
1.3 Derivation of Green’s functions

1.2.3 Diffraction gratings

An understanding of diffraction gratings is necessary in order to explain a number of wave scattering behaviours seen in this thesis. This includes the phenomenon of Wood anomalies, as discussed below [13]. Diffraction gratings are also useful in obtaining the solution to doubly periodic array problems, as outlined later in Chapter 5 for a square lattice, as any regular lattice (i.e., a square or hexagonal array) can be regarded as an infinite stack of diffraction gratings. Here a platonic diffraction grating is defined as a regularly spaced arrangement of scatterers which reflect and refract flexural wave energy according to the Fraunhofer grating equation, which is given by

$$\alpha_m = \alpha_0 + \frac{2m\pi}{d}, \quad (1.2.17)$$

where $$\alpha_m = k \sin \theta_m$$, $$\alpha_0 = k \sin \theta_i$$, and $$d$$ is the grating period [13]. Here $$\theta_i$$ represents the incident angle and $$\theta_m$$ the angle of propagation for the $$m$$th specular order. For any propagating incident wave we are guaranteed that the zeroth order will be present (as the grating acts like a mirror for $$m = 0$$), and we can determine whether additional plane waves are excited by computing

$$\chi_m = \begin{cases} \sqrt{k^2 - \alpha_m^2}, & \text{if } |\alpha_m| \leq k, \\ i\sqrt{\alpha_m^2 - k^2}, & \text{if } |\alpha_m| > k. \end{cases} \quad (1.2.18)$$

Provided $$\chi_m \in \mathbb{R}$$ then the corresponding specular order $$m$$ is propagating, otherwise it is an evanescent (non-propagating) wave. Evanescent waves do not transport energy to infinity, instead they decay exponentially as one moves in directions away from the grating.

In Figure 1.5 we provide an example to demonstrate the grating equation geometrically (using the values $$k = 5$$ and $$\theta_i = 30^\circ$$); here the 0th, 1st, and 2nd orders are propagating, while all other integer orders are evanescent. Particular combinations of incident angle and wave number will sometimes give rise to $$\chi_m \equiv 0$$ which corresponds to a Wood anomaly, where the excited wave travels along the grating interface to infinity [13, 81].

1.3 Derivation of Green’s functions

In this section we provide a brief derivation of the Green’s functions (GFs) which are used throughout this thesis. We begin with a brief derivation of the GF for the Helmholtz equation, and then extend the final result to obtain the GFs for the modified Helmholtz equation and biharmonic equation cases. The GF expression for the biharmonic plate equation is especially relevant in Chapters 3 and 4, as it is used to obtain the solution to several pinned plate problems.
1.3 Derivation of Green’s functions

1.3.1 The free-space Green’s function for the Helmholtz equation in two dimensions

We begin by examining the GF for the Helmholtz equation in Cartesian coordinates, which satisfies the following relation

\[(\Delta + k^2) G^H = \delta(x - x')\delta(y - y'),\]  \hspace{1cm} (1.3.1)

where \(G^H\) is our Green’s function, \(\Delta = \partial_x^2 + \partial_y^2\) denotes the Laplacian and \(\delta(z)\) represents the Dirac delta function. Here \((x, y)\) denotes the location of the field point, and \((x', y')\) the location of the source point. In this section we aim to derive the free-space Green’s function in terms of polar coordinates, where \((x, y) = (r \cos \theta, r \sin \theta)\). Subsequently the Laplacian is given by

\[\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2,\]  \hspace{1cm} (1.3.2)

and for the Dirac delta term, a change of coordinates from \(x\) to \(\zeta\) requires the use of the identity \[18\]

\[\delta(x - x') = \frac{1}{|\det J|} \delta(\zeta - \zeta'),\]  \hspace{1cm} (1.3.3)

where \(J\) denotes the Jacobian. Accordingly for our polar coordinate system we write

\[\delta(x - x')\delta(y - y') = \frac{1}{r} \delta(r - r')\delta(\theta - \theta').\]  \hspace{1cm} (1.3.4)

It is easy to see that this additional \(1/r\) factor is necessary by considering the identity

\[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x')\delta(y - y')dx dy = 1,\]  \hspace{1cm} (1.3.5a)
or equivalently
\[
\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{r} \delta(r - r') \delta(\theta - \theta') r dr d\theta = 1.
\] (1.3.5b)

Subsequently the two-dimensional Helmholtz equation in polar coordinates takes the form
\[
\left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_{\theta} + k^2 \right) G^H(r) = \frac{1}{r} \delta(r - r') \delta(\theta - \theta').
\] (1.3.6)

Since we are considering the free-space Green’s function, we can assume that there is no angular dependence (i.e., the Green’s function is independent of \(\theta\)). To this end, we integrate our PDE (1.3.6) with respect to \(\theta\), where
\[
\int_{\theta=0}^{2\pi} \delta(\theta - \theta') d\theta = 1.
\] (1.3.8)

Subsequently, we obtain the radially dependent form
\[
\left( \partial_r^2 + \frac{1}{r} \partial_r + k^2 \right) G^H(r) = \frac{\delta(r - r')}{2\pi r}.
\] (1.3.9)

In the absence of our forcing term, the PDE (1.3.9) above is known as Bessel’s equation [157] which has homogeneous solutions of the form
\[
G^H(r) = AH_0^{(1)}(kr) + BH_0^{(2)}(kr),
\] (1.3.10)

where \(H_0^{(1)}(z)\) and \(H_0^{(2)}(z)\) denote Hankel functions of the first and second kind, respectively.

We then impose a radiation condition to ensure that only outgoing waves are present at infinity (that is, we remove components of the homogeneous solution (1.3.10) which correspond to incoming waves at infinity, which are non-physical). Subsequently we impose the two-dimensional Sommerfeld radiation condition [80]
\[
\lim_{r \to \infty} \sqrt{r} (\partial_r - ik) G^H = 0,
\] (1.3.11)

to obtain
\[
\lim_{r \to \infty} \left\{ -\sqrt{r} A \left[ H_1^{(1)}(kr) + iH_0^{(1)}(kr) \right] - \sqrt{r} B \left[ H_1^{(2)}(kr) + iH_0^{(2)}(kr) \right] \right\} = 0.
\] (1.3.12)

It is straightforward to show that
\[
\lim_{r \to \infty} \sqrt{r} \left[ H_1^{(1)}(kr) + iH_0^{(1)}(kr) \right] = 0,
\] (1.3.13)

and since the second limit in (1.3.12) is unbounded as \(r \to \infty\), therefore \(B \equiv 0\). Next we examine
the behaviour of the GF as \( r \to 0 \). This is done by integrating our PDE \((1.3.9)\) over a circle of radius \( \varepsilon \) (which is denoted by \( \Omega_\varepsilon \)) and examining the behaviour as \( \varepsilon \to 0 \). This yields

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{\delta(r)}{2\pi r} \, dV = \lim_{\varepsilon \to 0} \int_{r=0}^{\varepsilon} \int_{\theta=0}^{2\pi} \frac{\delta(r)}{2\pi} \, dr \, d\theta = 1, \quad (1.3.14a)
\]

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} k^2 G^H \, dV = \lim_{\varepsilon \to 0} k^2 \int_{r=0}^{\varepsilon} \int_{\theta=0}^{2\pi} r H_0^{(1)}(kr) \, dr \, d\theta = 0. \quad (1.3.14b)
\]

For the Laplacian term one can use the Divergence theorem \([47]\) to write

\[
\int_{\Omega_\varepsilon} \Delta G^H \, dV = \int_{\partial \Omega_\varepsilon} \nabla \cdot \nabla G^H \, dV = \int_{\partial \Omega_\varepsilon} \nabla G^H \cdot n \, dS = \int_{\partial \Omega_\varepsilon} \partial_r G^H \cdot n \, dS. \quad (1.3.15)
\]

This result extends to our radially dependent form of the Laplacian \((1.3.9)\) as \( G \) is independent of \( \theta \). Returning to our homogeneous solution we have \( G^H(r) = AH^0(1)(kr) \) with a corresponding derivative \( \partial_r G^H(r) = -Ak H_1^{(1)}(kr) \). Consequently

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Delta G^H \, dV = \lim_{\varepsilon \to 0} \int_{r=0}^{\varepsilon} \int_{\theta=0}^{2\pi} \left\{ r \partial_r G^H \right\} \bigg|_{r=\varepsilon} \, d\theta = -2\pi Ak \lim_{\varepsilon \to 0} \left\{ \varepsilon H_1^{(1)}(\varepsilon k) \right\} = 4Ai. \quad (1.3.16)
\]

Combining \((1.3.14a)\), \((1.3.14b)\) and \((1.3.16)\), one can deduce that \( A = -i/4 \) and that the free-space Green’s function takes the form

\[
G^H(r) = -\frac{i}{4} H^0_0(ikr). \quad (1.3.17)
\]

### 1.3.2 The free-space Green’s function for the modified Helmholtz operator in two dimensions

For the case of the modified Helmholtz operator

\[
(\Delta - k^2)w(x) = 0, \quad (1.3.18)
\]

the Green’s function satisfies

\[
(\Delta - k^2)G^M = \delta(x - x'), \quad (1.3.19)
\]

and can be obtained directly by replacing \( k \) with \( ik \) in \((1.3.17)\) to obtain

\[
G^M(r) = -\frac{i}{4} H^0_1(ikr) = -\frac{1}{2\pi} K_0(kr). \quad (1.3.20)
\]

This final representation follows from the Bessel function relation \((9.6.4)\) in Abramowitz and Stegun \([1]\), and is given by

\[
K_0(z) = \frac{\pi i}{2} H^0_0(iz). \quad (1.3.21)
\]
1.3.3 The free-space Green’s function for the biharmonic plate equation

For the case of the biharmonic plate equation the Green’s function satisfies

\[(\Delta - k^2)(\Delta + k^2)G^P = \delta(x - x').\]  \(1.3.22\)

If we presume that the Green’s function can be written in the form \(G^P = \alpha_1 G^H + \alpha_2 G^M\), one can obtain

\[(\Delta - k^2)(\Delta + k^2) \{\alpha_1 G^H + \alpha_2 G^M\} = \alpha_1 (\Delta - k^2)\delta(x - x') + \alpha_2 (\Delta + k^2)\delta(x - x').\]  \(1.3.23\)

This gives rise to the system

\[
\begin{align*}
\alpha_1 + \alpha_2 &= 0, \\
-k^2\alpha_1 + k^2\alpha_2 &= 0,
\end{align*}
\]

which has the solution \(\alpha_2 = -\alpha_1 = 1/(2k^2)\). Accordingly,

\[
G^P(r) = \frac{i}{8k^2} \left[ H_0^{(1)}(kr) - H_0^{(1)}(ikr) \right] = \frac{1}{4k^2\pi} \left[ K_0(ikr) - K_0(kr) \right] \\
= \frac{i}{8k^2} \left[ H_0^{(1)}(kr) - \frac{2}{\pi i} K_0(kr) \right].
\]  \(1.3.25\)

Note that an identical result can be obtained via the procedure outlined in Boyling [17], which applies to the more general case of computing Green’s functions for arbitrary polynomials in the Laplacian.

1.4 Boundary element methods

The boundary element method (BEM) is a powerful technique for solving certain partial differential equations (PDEs). It has several advantages compared to finite element methods (FEMs) including reductions in complexity and computational time. This is achieved as only a solution at the boundary is required, which can be obtained by solving small systems of non-sparse matrices. The BEM is used here to obtain the numerical solution to problems involving arbitrarily shaped geometries, as seen in Chapter 5.

BEMs can be applied to various PDEs including Laplace’s equation, Poisson’s equation, Helmholtz’s equation and the biharmonic plate equation, but they are not as widely applicable as FEMs, as BEMs rely on the linearity of the governing PDE.

We outline the constant-panel BEM here based on the work by Wrobel and Aliabadi [161] and Liu [82], and construct a solution to the Helmholtz equation as a motivating example.
1.4 Boundary element methods

1.4.1 Boundary integral equations

In order to construct boundary element solutions to PDEs it is first necessary to determine the appropriate boundary integral equations.

Here we consider the problem of plane incident wave scattering by an arbitrarily shaped scatterer, where the external domain is governed by the Helmholtz equation

\[
(\Delta + k^2) w^H = 0.
\]  

(1.4.1)

Since the Helmholtz equation is a linear PDE we can decompose the wave function \( w^H \) into incident and scattered components \( w^H = w^H_i + w^H_s \) and consider Green’s second identity [166]

\[
\int_{\Omega_c} \{u\Delta v - v\Delta u\} \, dV = \int_{\partial\Omega} \{u\partial_{n'} v - v\partial_{n'} u\} \, dS'.
\]  

(1.4.2)

Provided that \( v \) is always defined as a Green’s function, for example \( v = G^H \) from (1.3.17), this can be expressed as [166]

\[
\int_{\partial\Omega} \{u(x')\partial_{n'}G^H(x,x') - \partial_{n'}u(x')G^H(x,x')\} \, dS' = \begin{cases} 
\quad u(x) & \text{if } x \in \Omega_\infty \\
\frac{1}{2} u(x) & \text{if } x \in \partial\Omega \\
\quad 0 & \text{if } x \in \Omega_c
\end{cases},
\]  

(1.4.3)

where \( x \) represents the location of the field point, \( \partial_{n'} \) denotes the normal derivative taken with respect to the source point \( x' \), \( \Omega_c \) represents our scatterer with an associated boundary \( \partial\Omega \), surrounded by an infinite domain \( \Omega_\infty \). The 1/2 term in (1.4.3) arises when \( x \in \partial\Omega \) and subsequently the integral exists only in a Cauchy Principal Value sense. Note that \( G^H(x,x') = G^H(x',x) \) and so one can easily express this relation in an alternative form by simply swapping the field and source points. An outline of our scattering problem is given in Figure 1.6 below.
By specifying \( u = w^H_s \) it is then possible to obtain

\[
\frac{1}{2} w^H_s(x) = \int_{\partial \Omega} \left\{ w^H_s(x') \partial_{n'} G^H(x,x') - G^H(x,x') \partial_{n'} w^H_i(x') \right\} \mathrm{d}S',
\]  

(1.4.4)

where the field point is restricted to the boundary \( \partial \Omega \), i.e. \( x \in \partial \Omega \).

The expression (1.4.4) can then be expressed in terms of the full potential \( w^H = w^H - w^H_i \) to obtain

\[
\frac{1}{2} w^H = \frac{1}{2} w^H_i + \int_{\partial \Omega} \left\{ w^H \partial_{n'} G^H - G^H \partial_{n'} w^H_i \right\} \mathrm{d}S' - \int_{\partial \Omega} \left\{ w^H_i \partial_{n'} G^H - G^H \partial_{n'} w^H_i \right\} \mathrm{d}S'.
\]  

(1.4.5)

Using Green’s second identity (1.4.2) once more with \( u = w^H_i \) and \( v = G^H \) we obtain an expression which now involves a nonvanishing boundary integral at infinity

\[
\frac{1}{2} w^H_i = \int_{\partial \Omega} \left\{ w^H_i \partial_{n'} G^H - G^H \partial_{n'} w^H_i \right\} \mathrm{d}S' + \int_{\partial \Omega_{\infty}} \left\{ w^H_i \partial_{n'} G^H - G^H \partial_{n'} w^H_i \right\} \mathrm{d}S',
\]  

(1.4.6a)

where \( x \in \partial \Omega \). Since \( x \in \partial \Omega \) is not contained in the region between \( \partial \Omega \) and \( \partial \Omega_{\infty} \), for the second integral above we have

\[
w^H_i = \int_{\partial \Omega_{\infty}} \left\{ w^H_i \partial_{n'} G^H - G^H \partial_{n'} w^H_i \right\} \mathrm{d}S',
\]  

(1.4.6b)

where \( x \in \partial \Omega \). Subsequently it can be established that

\[
\int_{\partial \Omega} \left\{ w^H_i \partial_{n'} G^H - G^H \partial_{n'} w^H_i \right\} \mathrm{d}S' = -\frac{1}{2} w^H_i,
\]  

(1.4.6c)

for \( x \in \partial \Omega \). This admits the boundary integral equation

\[
\frac{1}{2} w^H(x) = w^H_i(x) + \int_{\partial \Omega} \left\{ w^H(x') \partial_{n'} G^H(x,x') - G^H(x,x') \partial_{n'} w^H(x') \right\} \mathrm{d}S',
\]  

(1.4.7)

where \( x \in \partial \Omega \). Using this boundary integral equation it is then possible to determine the potential field \( w^H \) for an incident plane wave \( w^H_i \) which is scattered by an arbitrary structure \( \Omega_c \). This is achieved using boundary element methods.

### 1.4.2 An introduction to boundary element methods

Having determined the necessary boundary integral equations one can then impose boundary conditions at the edge of the scatterer. For example, we could specify Neumann boundary conditions

\[
\partial_{n'} w^H |_{x' \in \partial \Omega} = 0,
\]  

(1.4.8)
and then solve the system
\[ \frac{1}{2} w^H(x) = w^H_i(x) + \int_{\partial \Omega} w^H(x') \partial_{n'} G^H(x, x') dS', \tag{1.4.9} \]
for \( x \in \partial \Omega \). This is achieved by first discretising the boundary of our scatterer \( \partial \Omega \) into a finite number of panels (of approximately constant length) as seen in Figure 1.7, where we denote the \( q^{th} \) panel by \( \partial \Omega_q \) with corresponding midpoint \( \bar{x}_q \).

We then cycle through all possible field point terms which are evaluated at the panel midpoints \( \bar{x}_p \), and cycle through all possible source point terms where across each panel one assumes that the potential is constant over that panel, i.e.:
\[ \int_{\partial \Omega_q} w^H(x') \partial_{n'} G^H(x, x') dS' \simeq w^H_i(\bar{x}_p) \int_{\partial \Omega_q} \partial_{n'} G^H(x, x') dS'. \tag{1.4.10} \]

This admits the matrix representation
\[ \frac{1}{2} \vec{\phi} = \vec{\phi}^I + H \vec{\phi}, \tag{1.4.11} \]
where we define the vectors \( \vec{\phi}_p = w^H_i(\bar{x}_p) \) and \( \vec{\phi}^I_p = w^H_i(\bar{x}_p) \) along with the matrix
\[ H_{pq} = \int_{\partial \Omega_q} \partial_{n'} G^H(\bar{x}_p, x') dS'. \tag{1.4.12} \]
This can then be solved by computing
\[ \vec{\phi} = \left( \frac{1}{2} I - H \right)^{-1} \vec{\phi}^I, \tag{1.4.13} \]
where \( I \) denotes the identity matrix, to recover the unknown potential at the boundary. Similarly one could impose Dirichlet boundary conditions
\[ w^H(x') \bigg|_{x' \in \partial \Omega} = 0, \tag{1.4.14} \]
assume that the normal derivative of the potential is constant over each panel, i.e.:
\[ \int_{\partial \Omega_q} \partial_{n'} w^H(x') G^H(x, x') dS' \simeq \partial_{n'} w^H(x') \int_{\partial \Omega_q} G^H(x, x') dS', \tag{1.4.15} \]
and then solve the matrix system

$$\vec{\psi} = G^{-1}\vec{\varphi}. \quad (1.4.16)$$

This then allows us to recover the unknown normal derivative of the potential at the boundary, where $$\vec{\psi}_p = \partial_{n'}w^H(\bar{x}_p)$$ and

$$G_{pq} = \int_{\partial\Omega_q} G^H(\bar{x}_p, x')dS'. \quad (1.4.17)$$

One can then use the edge solution and the associated boundary condition to reconstruct the solution in external domain via

$$w^H(x) = w^H_i(x) + \int_{\partial\Omega} \left\{ w^H(x')\partial_{n'}G^H(x, x') - G^H(x, x')\partial_{n'}w^H(x') \right\} dS', \quad (1.4.18)$$

where $$x \in \Omega_\infty$$.

Note that one particular advantage of using BEMs is that we can easily implement different boundary conditions and solve the associated system straightforwardly. The biggest difficulty is in the numerical evaluation of the matrices $$H$$ and $$G$$.

### 1.4.3 Evaluation of $$G$$ and $$H$$

Having previously discretised our boundary $$\partial\Omega$$ we now consider the numerical evaluation of the matrices $$H$$ and $$G$$.

For an arbitrary panel $$\partial\Omega_q$$ we have endpoints $$(x', y') = (a, b)$$ and $$(x', y') = (c, d)$$, with panel length $$v_q = \sqrt{(a-c)^2 + (b-d)^2}$$. Accordingly we can parameterise the panel as the straight line

$$x'(t) = \left( \frac{c-a}{2} \right) t + \left( \frac{a+c}{2} \right), \quad (1.4.19a)$$

$$y'(t) = \left( \frac{d-b}{2} \right) t + \left( \frac{b+d}{2} \right), \quad (1.4.19b)$$

for $$t \in (-1, 1)$$. This panel $$\partial\Omega_q$$ has a normalised normal vector $$\mathbf{n}' = \frac{1}{v_q} (b-d, c-a)$$ which is taken to be pointing outwards from the boundary $$\partial\Omega$$ into the region $$\Omega_\infty$$. We now consider the matrix

$$G_{pq} = \int_{\partial\Omega_q} G^H(\bar{x}_p, x')dS'$$

$$= \frac{iv_q}{8} \int_{-1}^{1} H_0^{(1)}(kR(\bar{x}_p; t) \right) dt, \quad (1.4.20)$$
where \( \mathbf{x}_p = (x_p, y_p) \) denotes the midpoint of the \( p^{th} \) panel,

\[
R(\mathbf{x}_p; t) = \sqrt{\left( x_p - \left( \frac{c - a}{2} \right) t - \left( \frac{a + c}{2} \right) \right)^2 + \left( y_p - \left( \frac{d - b}{2} \right) t - \left( \frac{b + d}{2} \right) \right)^2},
\]

and we have accounted for arc length \( dS' = \|\partial_t r\| \, dt \), where \( r = (x'(t), y'(t)) \).

It is then possible to evaluate (1.4.20) numerically using a quadrature scheme such as the midpoint method

\[
\int_a^b f(x) \, dx = (b - a) f \left( \frac{a + b}{2} \right)
\]

to reveal

\[
G_{pq} = \frac{i \nu_q}{4} H_0^{(1)}(kR(\mathbf{x}_p; 0)).
\]

(1.4.22)

Similarly for the matrix involving the normal derivative of the Green’s function we have

\[
\mathbf{H}_{pq} = \int_{\partial \Omega_q} \left\{ \left( \partial_x G^H(\mathbf{x}_p, \mathbf{x}'), \partial_y G^H(\mathbf{x}_p, \mathbf{x}') \right) \cdot \mathbf{n}' \right\} \, dS'
\]

\[
= \frac{ik}{8} \int_{-1}^{1} \frac{H_1^{(1)}(kR(\mathbf{x}_p; t))}{R(\mathbf{x}_p; t)} S(\mathbf{x}_p; t) \, dt,
\]

(1.4.23)

where

\[
S(\mathbf{x}_p; t) = \left\{ x_p - \left( \frac{c - a}{2} \right) t - \left( \frac{a + c}{2} \right) \right\} (b - d) + \left\{ y_p - \left( \frac{d - b}{2} \right) t - \left( \frac{b + d}{2} \right) \right\} (c - a).
\]

(1.4.24)

Note that there is no \( \nu_q \) term present in (1.4.23) due to the scaling of the normal vector. Using the midpoint method we can obtain the representation

\[
\mathbf{H}_{pq} = \frac{ik}{4} S(\mathbf{x}_p; 0) H_1^{(1)}(kR(\mathbf{x}_p; 0)).
\]

(1.4.25)

The only difficulty that remains is when \( \mathbf{x}_p = \mathbf{x}_q \), i.e. \( r \equiv 0 \), where the matrix \( \mathbf{G} \) becomes singular along the diagonal. This can be seen clearly by inspecting (1.4.22) and observing that \( H_0^{(1)}(0) = -i \infty \). However, for the \( \mathbf{H} \) matrix expression (1.4.23) under midpoint rule quadrature, we see that \( S(\mathbf{x}_q; t) \equiv 0 \) and consequently one can specify

\[
\mathbf{H}_{qq} = 0.
\]

(1.4.26)

### 1.4.4 The singularity subtraction method

There are several established techniques for dealing with singular boundary integrals [82, 161]. The simplest of these techniques is to specify \( \mathbf{G}_{qq} = 0 \) and discretise with a large number of panels to reduce the influence of these diagonal terms on the solution.
A more sophisticated technique is the singularity subtraction method, which involves an understanding of the asymptotic behaviour of the Green’s function as $\hat{x}_p \rightarrow \hat{x}_q$. In our particular problem as $\hat{x}_p \rightarrow \hat{x}_q$ the expression (1.4.20) takes the form

$$G_{qq} = \frac{i\nu_q}{8} \int_{-1}^{1} H_0^{(1)} \left( \frac{k\nu_q|t|}{2} \right) \, dt, \quad (1.4.27)$$

which has a logarithmic singularity as $t \rightarrow 0$, and as such cannot be evaluated directly. However we can add and subtract an appropriately weighted logarithmic function to obtain

$$G_{qq} = \frac{\nu_q}{2} \int_{-1}^{1} \left\{ \frac{i}{4} H_0^{(1)} \left( \frac{k\nu_q|t|}{2} \right) + \frac{1}{2\pi} \log \left( \frac{k\nu_q|t|}{2} \right) \right\} \, dt - \frac{\nu_q}{2} \int_{-1}^{1} \frac{1}{2\pi} \log \left( \frac{k\nu_q|t|}{2} \right) \, dt. \quad (1.4.28)$$

Here the first integral is now regular as $|t| \rightarrow 0$ and the second integral can be evaluated analytically. Using the midpoint rule we can determine that the first integral can be expressed as

$$\frac{\nu_q}{2} \int_{-1}^{1} \left\{ \frac{i}{4} H_0^{(1)} \left( \frac{k\nu_q|t|}{2} \right) + \frac{1}{2\pi} \log \left( \frac{k\nu_q|t|}{2} \right) \right\} \, dt = \frac{\nu_q}{2} \left[ \frac{-2\gamma_e + i\pi + \log(4)}{4\pi} \right], \quad (1.4.29)$$

(which considers the limit as $|t| \rightarrow 0$) with $\gamma_e = 0.577215\ldots$ representing the Euler–Mascheroni constant. We can evaluate the second integral analytically to establish that

$$-\frac{\nu_q}{4\pi} \int_{-1}^{1} \log \left( \frac{k\nu_q|t|}{2} \right) \, dt = -\frac{1}{k\pi} \int_{0}^{k\nu_q/2} \log(\tau) \, d\tau = \frac{\nu_q}{2\pi} \left[ 1 - \log \left( \frac{k\nu_q}{2} \right) \right], \quad (1.4.30)$$

which follows from a simple change of variables

$$\int_{-1}^{-1} \log(|t|) \, dt = 2 \int_{0}^{1} \log(\tau) \, d\tau. \quad (1.4.31)$$

Subsequently the following analytical expression for the diagonal terms of our first matrix can obtained in the form

$$G_{qq} = \frac{\nu_q}{2\pi} \left[ -\gamma_e + 1 + \frac{i\pi}{2} + \log \left( \frac{4}{k\nu_q} \right) \right], \quad (1.4.32)$$

which depends entirely on the approximately constant length of each panel $\nu_q$.

Using the singularity subtraction expressions (1.4.32) for the diagonal terms as opposed to merely imposing a vanishing condition enables much faster convergence to the edge solution, for low panel discretisations of the boundary.
1.5 Discussion

We have demonstrated here a number of background concepts which are highly relied upon in the subsequent chapters of this thesis. These include derivations of the EB beam and KL plate equations, an introduction to reciprocal spaces, diffraction gratings, and constant-panel BEMs. We proceed to an outline of the different PlaCs which can be fabricated using circular inclusions, which is the topic of Chapter 2.
2

Multipole solutions for circular inclusions in elastic plates

“Surely we can write a paper titled ‘Slowing Platonic Love waves’ ”
Ross McPhedran, in reference to Kirchoff–Love waves

2.1 Introduction

In this chapter we outline the solution to a number of scattering problems involving circular geometries in the setting of the biharmonic plate equation. A number of results and procedures shown here are also applicable to pinned geometries and thus form an important part of this thesis. Furthermore, the solutions to problems involving circular geometries also act as a useful verification of BEM code, which is used to obtain the solution for geometries of arbitrary shape in Chapter 5. Here we outline the solution to the problems of a single circular scatterer, an infinite one-dimensional array of circles (or grating), and a doubly periodic square array of circular bodies. At the edge of each body we impose clamped or free-edge boundary conditions. An outline of the three problems considered is shown in Figure 2.1.

An understanding of circular geometries is of considerable interest, as analytical solutions can be found using multipole techniques. Geometries such as squares have been considered by Farhat et al. [41], but to the best of the author’s knowledge, the extension to arbitrarily shaped
scatters in thin plates is considered only by Smith et al. [135, 138]. There are only a select few geometries where closed-form solutions are possible, such as circles and ellipses, and even then efficient numerical evaluation can be a problem. For example, the solution for elliptical scatterers can be expressed in terms of Mathieu functions which are notoriously difficult to evaluate numerically.

The solution for the problem of wave scattering by a single circular scatterer is well known and can be found in Konenkov [67], Leissa [73], Norris and Vemula [111]. The extension to wave scattering by an infinite array (or grating) of circular cylinders is outlined in Movchan et al. [107] and the problem of Bloch–Floquet waves through a doubly periodic array of circular inclusions is considered in Movchan et al. [106] and Poulton et al. [124]. We provide an outline of the solution procedures used in these publications and employ these procedures at a later stage to arbitrary geometries and scattering by pins.

We begin with the frequency domain representation of the biharmonic plate equation in two dimensions, which is given by

\[
(\Delta^2 - k^4)w(x) = (\Delta + k^2)(\Delta - k^2)w(x) = 0,
\]

where \(x = (x, y)\), \(k^2 = \omega \sqrt{\rho h / D}\) is the non-dimensionalised wave number, \(\omega\) denotes the angular
2.2 Single circular scatterer

frequency. Here $\rho$ denotes the mass density, $h$ the thickness, and $D$ the flexural rigidity of the plate. The solution to the biharmonic plate equation can then be decomposed into Helmholtz and modified Helmholtz components ($w = w^H + w^M$) with corresponding multipole solutions

\[ w^H(x) = \sum_{n=-\infty}^{\infty} \left[ A_n J_n(kr) + B_n H_n^{(1)}(kr) \right] e^{in\theta}, \quad (2.1.2a) \]

and

\[ w^M(x) = \sum_{n=-\infty}^{\infty} \left[ C_n I_n(kr) + D_n K_n(kr) \right] e^{in\theta}, \quad (2.1.2b) \]

where $J_n$, $I_n$, and $K_n$ are Bessel functions of the first and second kind and $H_n^{(1)}$ is a Hankel function of the first kind. Here we use the polar coordinate system $(x, y) = (r \cos \theta, r \sin \theta)$. Note that for (2.1.2a) we could alternatively use the Bessel function $Y_n$ in place of the Hankel function $H_n^{(1)}$. Furthermore, it is also possible to express $I_n$ and $K_n$ in terms of $J_n$ and $H_n^{(1)}$ via the relations

\[ I_n(z) = i^{-n} J_n(iz), \quad (2.1.3a) \]

\[ H_n^{(1)}(iz) = \frac{2}{\pi i^{n+1}} K_n(z), \quad (2.1.3b) \]

as given by (9.6.3) and (9.6.4) of Abramowitz and Stegun [1]. Due to the linearity of our partial differential equation we can construct the solution to the biharmonic plate equation as the following superposition

\[ w(x) = \sum_{n=-\infty}^{\infty} \left[ A_n J_n(kr) + B_n H_n^{(1)}(kr) + C_n I_n(kr) + D_n K_n(kr) \right] e^{in\theta}. \quad (2.1.4) \]

For an external scattering problem (i.e., an infinitely large plate containing a single circular inclusion) a solution can be obtained analytically using (2.1.4) provided we remove components of the solution which do not satisfy the radiation condition as $r \to \infty$. Similarly for the free vibration of a finite plate, the singular Bessel function components of the solution must be removed in order to obtain a solution [73].

We now consider the specific case when plane waves are incident upon a single circular scatterer in an infinite domain.

2.2 Single circular scatterer

The displacement of the plate in the exterior region to the scatterer can be computed simply for the case of a circular cavity. This is outlined by Konenkov [67], Norris and Vemula [111], who
decompose the solution for a circle of radius $a$ into incident and scattered fields as follows

$$w(x) = w_i^H(x) + \sum_{n=-\infty}^{\infty} \left[ A_n H_n^{(1)}(kr) + B_n H_n^{(1)}(ikr) \right] e^{in\theta}. \tag{2.2.1}$$

A standard Helmholtz incident plane wave is then considered, and subsequently represented in terms of cylindrical waves via the Jacobi–Anger identity [22]

$$w_i^H(x) = e^{ikx} = e^{ikr \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\theta}. \tag{2.2.2}$$

It is mathematically possible to construct an incident wave corresponding to the modified Helmholtz field, but this is non-physical for the case of a single scatterer. After applying the relevant boundary conditions at $r = a$, it is a straightforward task to obtain a system for the unknown coefficients in equation (2.2.1). We demonstrate this for the case of clamped-edge boundary conditions which are given by

$$w|_{r=a} = 0, \tag{2.2.3a}$$
$$\partial_r w|_{r=a} = 0. \tag{2.2.3b}$$

Applying these conditions at $r = a$ yields

$$\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = -i^n \begin{bmatrix} e_n \\ f_n \end{bmatrix}, \tag{2.2.4}$$

where

$$a_n = \frac{k}{2} \left[ H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka) \right], \tag{2.2.5a}$$
$$b_n = \frac{ik}{2} \left[ H_{n-1}^{(1)}(ika) - H_{n+1}^{(1)}(ika) \right], \tag{2.2.5b}$$
$$c_n = H_n^{(1)}(ka), \tag{2.2.5c}$$
$$d_n = H_n^{(1)}(ika), \tag{2.2.5d}$$
$$e_n = \frac{k}{2} \left[ J_{n-1}(ka) - J_{n+1}(ka) \right], \tag{2.2.5e}$$
$$f_n = J_n(ka). \tag{2.2.5f}$$

Similarly for free-edge boundary conditions, which are given by

$$\Delta w|_{r=a} = \frac{(1 - \nu)}{r} \left( \partial_r + \frac{1}{r} \partial_\theta^2 \right) w|_{r=a} = 0, \tag{2.2.6a}$$
$$\partial_r \Delta w|_{r=a} = \frac{(1 - \nu)}{r^2} \left( \frac{1}{r} - \partial_r \right) \partial_\theta^2 w|_{r=a} = 0. \tag{2.2.6b}$$
where $\nu$ denotes the Poisson ratio of the material, we obtain

\begin{align*}
a_n &= \left[ \frac{\beta n^2}{a^2} - k^2 \right] H_n^{(1)}(ka) - \frac{\beta k}{2a} \left[ H_{n+1}^{(1)}(ka) - H_{n-1}^{(1)}(ka) \right], \\
    b_n &= \left[ \frac{\beta n^2}{a^2} + k^2 \right] H_n^{(1)}(ika) - \frac{i\beta k}{2a} \left[ H_{n+1}^{(1)}(ika) - H_{n-1}^{(1)}(ika) \right], \\
c_n &= -\frac{k}{2} \left[ \frac{\beta n^2}{a^2} + k^2 \right] \left[ H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka) \right] + \frac{\beta n^2}{a^3} H_n^{(1)}(ka), \\
d_n &= -\frac{i k}{2} \left[ \frac{\beta n^2}{a^2} - k^2 \right] \left[ H_{n-1}^{(1)}(ika) - H_{n+1}^{(1)}(ika) \right] + \frac{\beta n^2}{a^3} H_n^{(1)}(ika), \\
e_n &= \left[ \frac{\beta n^2}{a^2} - k^2 \right] J_n(ka) - \frac{\beta k}{2a} \left[ J_{n-1}(ka) - J_{n+1}(ka) \right], \\
f_n &= -\frac{k}{2} \left[ \frac{\beta n^2}{a^2} + k^2 \right] \left[ J_{n-1}(ka) - J_{n+1}(ka) \right] + \frac{\beta n^2}{a^3} J_n(ka).
\end{align*}

Incidentally, one can also obtain the solution for the associated interior problem (the problem of plate vibration) by omitting terms that are singular as $kr \to 0$ and considering the expansion

\begin{equation}
w(x) = \sum_{n=-\infty}^{\infty} \left( A_n J_n(kr) + B_n I_n(kr) \right) e^{in\theta}.
\end{equation}

Applying the boundary conditions for the clamped case (2.2.3) or for the free-edge case (2.2.6b), as outlined in Leissa [73], yields a homogeneous system of equations which can be expressed in matrix form. It is then possible to obtain the frequencies of vibration for a clamped or free circular plate by searching for values of $k$ which are associated with vanishing determinant. The associated eigenvectors reveal the coefficients necessary to determine the associated mode of vibration, which has a simple $\exp(in\theta)$ dependence.

## 2.3 An infinite grating of circular scatterers

We now consider an infinite number of circular inclusions situated along the $x$-axis at $(x, y) = (md, 0)$ where $m \in \mathbb{Z}$ and $d$ denotes the period of the grating. This is shown in Figure 2.1(b). The theory for this particular problem is outlined in Movchan et al. [105, 107] and Linton [77] (for the Helmholtz equation) and is included here both for completeness and as a check on numerics.

We begin by constructing the quasiperiodic Green’s function associated with a single array of equally spaced cylinders.
2.3 An infinite grating of circular scatterers

2.3.1 Quasiperiodic Green’s functions for a grating

For the problem of plane wave scattering by an array, it is important to consider the quasiperiodic Green’s functions of Helmholtz and modified Helmholtz type. These GFs satisfy

\[
(\Delta + k^2) G^H_g(x, x') = -\delta(y - y') \sum_{m=-\infty}^{\infty} \delta(x - x' - md)e^{i\alpha_0 md}, \tag{2.3.1a}
\]

\[
(\Delta - k^2) G^M_g(x, x') = -\delta(y - y') \sum_{m=-\infty}^{\infty} \delta(x - x' - md)e^{i\alpha_0 md}, \tag{2.3.1b}
\]

as well as the quasiperiodicity conditions

\[
G^H_g(x + md, y; x') = G^H_g(x; x')e^{i\alpha_0 md}, \tag{2.3.2a}
\]

\[
G^M_g(x; x' + md, y') = G^M_g(x; x')e^{-i\alpha_0 md}, \tag{2.3.2b}
\]

where \(\alpha_0 = k \sin(\theta_i)\), for \(m \in \mathbb{Z}\) and \(\delta(z)\) denotes the Dirac delta function.

As with periodic functions, by applying a quasiperiodic condition we can reduce our attention down to a fundamental period \(-d/2 < x < d/2\) where the height in \(y\) is arbitrary. However as we aim to consider infinite stacks of gratings at a later stage, we consider the unit cell \((x, y) = [-d/2, d/2] \times [-d/2, d/2]\) for simplicity. Within this cell we can construct the representations

\[
G^H_g(x, x') = \frac{1}{4} \sum_{m=-\infty}^{\infty} H^{(1)}_0 \left( k \sqrt{(x - x' - md)^2 + (y - y')^2} \right) e^{i\alpha_0 md}, \tag{2.3.3a}
\]

\[
G^M_g(x, x') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} K_0 \left( k \sqrt{(x - x' - md)^2 + (y - y')^2} \right) e^{i\alpha_0 md}. \tag{2.3.3b}
\]

These expressions (2.3.3) can be rewritten using Graf’s addition theorem (formula (9.1.79) in Abramowitz and Stegun [1])

\[
C_\nu(u)e^{iux} = \sum_{l=-\infty}^{\infty} C_{l+\nu}(u)J_l(v)e^{i\alpha}, \quad \text{where} \quad |ve^{i\alpha}| < |u|, \tag{2.3.4}
\]

within the fundamental cell, where \(C_\nu\) denotes an arbitrary Bessel function. Here the lines \(u, v\) and \(w\) form a closed triangle with \(w = \sqrt{u^2 + v^2 - 2uv \cos \alpha}\), \(\alpha\) denoting the angle between the lines \(u\) and \(v\), and \(\chi\) denoting the angle between the lines \(u\) and \(w\).
2.3 An infinite grating of circular scatterers

\[ \alpha = \theta - \phi \]

\[ \gamma = \text{arg} \xi \]

\[ \pi - \gamma \]

\[ \phi = \text{arg} x' \]

\[ x' \]

\[ x \]

\[ \xi \]

\[ \beta \]

\[ \theta = \text{arg} x \]

\[ x = (x, y) \]

\[ x' = (x', y') \]

\[ \xi = x - x' \]

\[ \text{H}0(k\xi) + \sum_{l=\infty}^{l=\infty} S_{l}^{H,G} J_{l}(k\xi)e^{-il\gamma} \]

\[ 1/2 \pi \left[ K_{0}(k\xi) + \sum_{l=\infty}^{l=-\infty} S_{l}^{K,G} I_{l}(k\xi)e^{-il\gamma} \right] \]

where we have the grating sums

\[ S_{l}^{H,G}(\alpha_0, k, d) = \sum_{n\neq 0} H_{l}^{(1)}(|n|kd)e^{i\alpha_0 nd}e^{il\varphi_n} \]

\[ S_{l}^{K,G}(\alpha_0, k, d) = \sum_{n\neq 0} K_{l}(|n|kd)e^{i\alpha_0 nd}e^{il\varphi_n} \]

As before, we define the vectors \( x = (x, y) \), \( x' = (x', y') \), and \( \xi = x - x' \), which have norms \( r \), \( r' \) and \( \xi \), respectively, with associated angles \( \theta = \text{arg} x \), \( \phi = \text{arg} x' \) and \( \gamma = \text{arg} \xi \), respectively. Furthermore \( \varphi_n = \pi H(\pm n) \) where \( H(x) \) denotes a Heaviside function.

We observe from Figure 2.2 that \( \beta = \pi - \gamma + \phi \) and subsequently \( \gamma = \chi + \theta \). Accordingly (2.3.5a) can be expressed in the form

\[ G_{g}^{H}(x, x') = \frac{i}{4} \left[ H_{0}^{(1)}(k\xi) + \sum_{l=\infty}^{l=\infty} S_{l}^{H,G} (-1)^l J_{l}(k\xi)e^{il\chi}e^{il\theta} \right] \]

after using the Bessel function identity \( J_{-\nu}(z) = (-1)^{-\nu}J_{\nu}(z) \) (formula (9.1.5) in Abramowitz and Stegun [1]) and reversing the order of summation.

Figure 2.2: An geometrical outline of Graf’s addition theor em, showing the three vectors \( x, x' \) and \( \xi \) and several important angles.
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Applying Graf’s addition theorem once more (with \( u = r, v = r', w = \xi \)), where

\[
H_0^{(1)}(k\xi) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr)J_n(kr')e^{in(\theta-\phi)}, \tag{2.3.8a}
\]

\[
J_l(k\xi)e^{il\chi} = \sum_{m=-\infty}^{\infty} J_{m+l}(kr)J_m(kr')e^{im(\theta-\phi)}, \tag{2.3.8b}
\]

and introducing the change of variable \( n = l + m \) we can obtain the form

\[
G^H_g(x, x') = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr)J_n(kr')e^{in(\theta-\phi)}
+ \frac{i}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S_{m-n}^{H,G}(-1)^{n+m} J_n(kr)J_m(kr')e^{in\theta-im\phi}, \tag{2.3.9}
\]

after observing that \((-1)^{m+n} = (-1)^{m-n}\). By replacing \( k \) with \( ik \) one can directly obtain the equivalent representation for the Green’s function associated with the modified Helmholtz equation

\[
G^M_g(x, x') = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} K_n(kr)I_n(kr')e^{in(\theta-\phi)}
+ \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S_{m-n}^{K,G}(-1)^{n+m} I_n(kr)I_m(kr')e^{in\theta-im\phi}, \tag{2.3.10}
\]

which is obtained after using the relations (2.1.3) and

\[
S_l^{H,G}(ik, \alpha_0) = \frac{2}{\pi^{l+1}} S_l^{K,G}(k, \alpha_0). \tag{2.3.11}
\]

Using these quasiperiodic Green’s function expressions we can obtain an analytical solution to the grating problem for circular inclusions. However in practise this relies on deriving convergent expressions for the grating sums (2.3.6a) and (2.3.6b). This is due to the fact that (2.3.6a) is only conditionally convergent in its present form, whereas (2.3.6b) can be evaluated directly in most instances. A discussion on convergent expressions for these grating sums is given in Appendix A of this thesis for reference. The solution is found by first computing Rayleigh identities, which is the topic of the next section.
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2.3.2 Derivation of the Rayleigh identities for a grating

We begin by highlighting that the solution of the biharmonic plate equation within the fundamental cell can be represented as

\[ w(x) = \sum_{n=-\infty}^{\infty} \left( A_n J_n(kr) + B_n H_n^{(1)}(kr) + C_n I_n(kr) + D_n K_n(kr) \right) e^{in\theta}, \]

and that the displacement must also satisfy the quasiperiodic condition

\[ w(x + md, y) = w(x)e^{i\alpha md}. \]

We then impose boundary conditions at the edge of the circle, such as clamped (2.2.3) or free-edge conditions (2.2.6). However, with four unknown coefficients and only two boundary conditions, it is necessary to consider a reduction in unknown terms. This is achieved via the use of dynamic Rayleigh identities. These are found by imposing a quasiperiodic boundary around a single scatterer and integrating over the subsequent cell. By suitable expansion of the displacement and its normal derivative one can then connect the array problem with the single cylinder problem, which is in essence the definition of the Rayleigh identity. In other words, it incorporates information about the array geometry into our problem. The procedure outlined here is given in Movchan et al. [105], Poulton et al. [121] and begins by considering Green’s second identity within the fundamental cell

\[ \int_{\Omega} \{ u \Delta v - v \Delta u \} \, dV = \int_{\partial \Omega} \{ u \partial_n' v' - v \partial_n' u \} \, dS', \]  

where \( \Omega \) represents the area between the scatterer and the fundamental cell boundary, \( \Omega_c \) represents the area inside the circle, and \( \partial \Omega \) denotes the boundary of the scatterer. The edges of the fundamental cell are represented by \( \gamma^+, \gamma^-, \gamma^l \) and \( \gamma^r \) which represent the top, bottom, left- and right-hand sides respectively. This can be seen clearly in Figure 2.3 below.

In the Green’s identity expression (2.3.14a) we specify \( u = w \) to represent the displacement, \( v \) to be the quasiperiodic Green’s function \( G_g^{H,M} \) and from (1.4.3) we have

\[ \int_{\partial \Omega} \{ w(x')\partial_n'G_g^{H,M}(x,x') - \partial_n'w(x')G_g^{H,M}(x,x') \} \, dS' = \begin{cases} w(x) & \text{if } x \in \Omega \\ \frac{1}{2} w(x) & \text{if } x \in \partial \Omega \\ 0 & \text{if } x \in \Omega_c \end{cases}. \]

Accordingly one obtains

\[ w(x) = \int_{\partial \Omega} \{ \partial_n'G_g^{H,M}(x,x')w(x') - \partial_n'w(x')G_g^{H,M}(x,x') \} \, dS', \]

assuming \( x \in \Omega \).

Note that the quasiperiodicity of \( w \) compared to that of the Green’s functions \( G_g^{H,M} \) (with respect
2.3 An infinite grating of circular scatterers

\[ \Omega_c \quad \partial \Omega \quad \gamma^+ \quad \Omega \quad \gamma^- \]

Figure 2.3: An outline of the fundamental cell relative to a grating of circular cylinders.

to the source point, i.e. (2.3.2b)) ensures that the contribution from the integral over \( \gamma^+ \) cancels with that from \( \gamma^- \), and that the integral over \( \gamma^- \) cancels with the integral over \( \gamma^+ \). That is, there is no contribution from the edges of the Wigner–Seitz cell. We then introduce the Fourier series expansions

\[ w(x') = \sum_{l=-\infty}^{\infty} a_l e^{il\phi}, \quad (2.3.15a) \]
\[ \partial_r w(x') = \sum_{l=-\infty}^{\infty} b_l e^{il\phi}, \quad (2.3.15b) \]

for \( \phi = (-\pi, \pi) \), which reveals

\[ w(x) = \int_{-\pi}^{\pi} \left\{ \left[ \sum_{l=-\infty}^{\infty} a_l e^{il\phi} \right] \left[ \frac{ik}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr)J'_n(kr') e^{in(\theta-\phi)} \right] \\
+ \frac{ik}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S^{H_G}_{m-n} (-1)^{n+m} J_n(kr)J'_m(kr') e^{in\theta-im\phi} \right\} \left| \frac{i}{r'} a d\phi. \quad (2.3.16) \right. \]

This can be easily evaluated by taking note of the following integral identity

\[ \int_{-\pi}^{\pi} e^{il\phi-in\phi} d\phi = 2\pi \int_{-1/2}^{1/2} e^{i(l-n)2\pi t} dt = 2\pi \delta_{ln}, \quad (2.3.17) \]
which allows us to obtain

\[
\begin{align*}
  w(x) &= \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr) \left( \frac{\pi a}{2} \right) [a_n k J'_n(ka) - b_n J_n(ka)] e^{in\theta} \\
  &\quad + \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} S_{m-n}^{H,G} (-1)^{n+m} J_n(kr) \left( \frac{\pi a}{2} \right) [a_m k J'_m(ka) - b_m J_m(ka)] e^{in\theta}. 
\end{align*}
\]

(2.3.18)

If one compares this against the original multipole expansion (2.1.2a) then

\[
\begin{align*}
  B_n &= \left( \frac{\pi a}{2} \right) [a_n k J'_n(ka) - b_n J_n(ka)], \\
  A_n &= \sum_{m=-\infty}^{\infty} \left( \frac{\pi a}{2} \right) (-1)^{n+m} S_{m-n}^{H,G} [a_m k J'_m(ka) - b_m J_m(ka)],
\end{align*}
\]

(2.3.19a,b)

and ultimately we can establish that

\[
A_n = \sum_{m=-\infty}^{\infty} (-1)^{n+m} S_{m-n}^{H,G} B_m. 
\]

(2.3.20)

This result is known as the dynamic Rayleigh identity for the Helmholtz equation, associated with an infinite grating of cylinders. Employing this procedure for the modified Helmholtz Green’s function (2.3.10) and comparing against (2.1.2b) reveals the Rayleigh identity

\[
C_n = \sum_{m=-\infty}^{\infty} (-1)^{m} S_{m-n}^{K,G} D_m. 
\]

(2.3.21)

Subsequently we can consider solutions to the grating problem in the presence of incident waves.

### 2.3.3 Multipole solution for wave scattering by a grating

It is possible to consider incident waves of both Helmholtz and modified Helmholtz type for the grating problem

\[
\begin{align*}
  w^H_i(x) &= \frac{\delta_0}{\sqrt{\chi_0}} e^{i\alpha_0 x - i\chi_0 y} = \frac{\delta_0}{\sqrt{\chi_0}} \sum_{l=-\infty}^{\infty} i^l \left( \frac{\alpha_0 + i\chi_0}{k} \right)^l J_l(kr) e^{il\theta}, \\
  w^M_i(x) &= \frac{\tilde{\delta}_0}{\sqrt{\tilde{\chi}_0}} e^{i\alpha_0 x - i\tilde{\chi}_0 y} = \frac{\tilde{\delta}_0}{\sqrt{\tilde{\chi}_0}} \sum_{l=-\infty}^{\infty} i^l \left( \frac{\alpha_0 + i\tilde{\chi}_0}{k} \right)^l I_l(kr) e^{il\theta},
\end{align*}
\]

(2.3.22a,b)

which represent incident waves travelling in the direction \( y < 0 \), [1, 107]. Here \( \delta_0 \) denotes the scaled amplitude of a Helmholtz incident wave and \( \tilde{\delta}_0 \) denotes the scaled amplitude of a modified Helmholtz incident wave (which both feature a square root prefactor in order to simplify the conservation of energy relations for the plate, which are given in Chapter 5). Also, \( \alpha_0 = k \sin \theta_i \), \( \chi_0 = k \cos \theta_i \), and \( \alpha_0^2 + \chi_0^2 = -k^2 \) (i.e. \( \chi_0 = i\tau_0 \) where \( \tau_0 > 0 \)) where \( \theta_i \) is the incident angle. Typically only one incident wave type is considered at a given time (e.g., \( \delta_0 = 1 \) and \( \tilde{\delta}_0 = 0 \),
2.3 An infinite grating of circular scatterers

however one can send in any combination of these two waves into our system simultaneously.
To clarify, a Helmholtz incident wave is a propagating plane wave solution to the Helmholtz
equation, and a modified Helmholtz incident wave is the analogous (but evanescent) wave field
for the modified Helmholtz equation.

Accordingly, in the presence of incident waves, the solutions to the Helmholtz (2.1.2a) and
modified Helmholtz (2.1.2b) equations now take the form

\[ w_H(x) = w_i^H(x) + \sum_{n=-\infty}^{\infty} \left( A_n J_n(kr) + B_n H_n^{(1)}(kr) \right) e^{\imath n \theta}, \]
\[ w_M(x) = w_i^M(x) + \sum_{n=-\infty}^{\infty} \left( C_n I_n(kr) + D_n K_n(kr) \right) e^{\imath n \theta}. \]

If we use the multipole representations of these incident waves (2.3.22a) and (2.3.22b) above in
addition to the dynamic Rayleigh identities (2.3.20) and (2.3.21) these become

\[ w_H = \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} (-1)^{n+m} S_{m-n}^{H,G} B_m \right) \left[ \frac{\delta_0}{\sqrt{|\chi_0|}} \imath^n \left( \frac{\alpha_0 + \imath \chi_0}{k} \right)^n J_n(kr) \right. \]
\[ \left. + B_n H_n^{(1)}(kr) \right] e^{\imath n \theta}, \] \hspace{1cm} (2.3.24a)

\[ w_M = \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} (-1)^m S_{m-n}^{K,G} D_m \right) \left[ \frac{\tilde{\delta}_0}{\sqrt{|\tilde{\chi}_0|}} \imath^n \left( \frac{\alpha_0 + \imath \tilde{\chi}_0}{k} \right)^n I_n(kr) \right. \]
\[ \left. + D_n K_n(kr) \right] e^{\imath n \theta}, \] \hspace{1cm} (2.3.24b)

with the biharmonic equation solution taking the familiar form \( w = w_H + w_M \) (i.e., a superpo-
sition of the solutions above).

We now impose boundary conditions at the edge of the scatterers, and for simplicity we begin
with clamped-edge boundary conditions (2.2.3a) and (2.2.3) which admits the system

\[ \sum_{m=-\infty}^{\infty} \left[ (-1)^{n+m} S_{m-n}^{H,G} J_n(ka) + H_n^{(1)}(ka) \delta_{mn} \right] B_m \]
\[ + \sum_{m=-\infty}^{\infty} \left[ (-1)^m S_{m-n}^{K,G} I_n(ka) + K_n(ka) \delta_{mn} \right] D_m \]
\[ = - \frac{\delta_0}{\sqrt{|\chi_0|}} \imath^n \left( \frac{\alpha_0 + \imath \chi_0}{k} \right)^n J_n(ka) - \frac{\tilde{\delta}_0}{\sqrt{|\tilde{\chi}_0|}} \imath^n \left( \frac{\alpha_0 + \imath \tilde{\chi}_0}{k} \right)^n I_n(ka), \] \hspace{1cm} (2.3.25a)
and
\[
\sum_{m=-\infty}^{\infty} \left[ (-1)^{n+m} S_{m-n}^{H,G} J_n'(ka) + H_n^{(1)'}(ka) \delta_{mn} \right] B_m \\
+ \sum_{m=-\infty}^{\infty} \left[ (-1)^{m} S_{m-n}^{K,G} J_n'(ka) + K_n'(ka) \delta_{mn} \right] D_m \\
= - \frac{\delta_0}{\sqrt{\chi_0}} i^n \left( \frac{\alpha_0 + i\chi_0}{k} \right)^n J_n'(ka) - \frac{\hat{\delta}_0}{\sqrt{\chi_0}} i^n \left( \frac{\alpha_0 + i\hat{\chi}_0}{k} \right)^n I_n'(ka), \tag{2.3.25b}
\]

where prime notation denotes derivatives with respect to \( r \). This can be represented in the matrix form
\[
\begin{bmatrix} [a_{mn}] & [b_{mn}] \\ [c_{mn}] & [d_{mn}] \end{bmatrix} \begin{bmatrix} [B_n] \\ [D_n] \end{bmatrix} = -i^n \left( \begin{bmatrix} [c_n] \\ [f_n] \end{bmatrix} \right), \tag{2.3.26}
\]

where we use the notation \([\zeta_n]\) to denote vectors and \([\zeta_{mn}]\) to denote matrices, and
\[
\begin{align*}
a_{mn} &= (-1)^{n+m} S_{m-n}^{H,G} J_n(ka) + H_n^{(1)}(ka) \delta_{mn}, \tag{2.3.27a} \\
b_{mn} &= (-1)^{m} S_{m-n}^{K,G} J_n(ka) + K_n(ka) \delta_{mn}, \tag{2.3.27b} \\
c_{mn} &= (-1)^{n+m} S_{m-n}^{H,G} J_n'(ka) + H_n^{(1)'}(ka) \delta_{mn}, \tag{2.3.27c} \\
d_{mn} &= (-1)^{m} S_{m-n}^{K,G} J_n'(ka) + K_n'(ka) \delta_{mn}, \tag{2.3.27d} \\
e_n &= \frac{\delta_0}{\sqrt{\chi_0}} \left( \frac{\alpha_0 + i\chi_0}{k} \right)^n J_n(ka) + \frac{\hat{\delta}_0}{\sqrt{\chi_0}} \left( \frac{\alpha_0 + i\hat{\chi}_0}{k} \right)^n I_n(ka), \tag{2.3.27e} \\
f_n &= \frac{\delta_0}{\sqrt{\chi_0}} \left( \frac{\alpha_0 + i\chi_0}{k} \right)^n J_n'(ka) + \frac{\hat{\delta}_0}{\sqrt{\chi_0}} \left( \frac{\alpha_0 + i\hat{\chi}_0}{k} \right)^n I_n'(ka). \tag{2.3.27f}
\end{align*}
\]

In Poulton et al. [124] a multipole solution for the doubly periodic free-edge problem is given. These expressions only require a slight modification in order to obtain the equivalent grating problem system. Accordingly we impose the free-edge conditions (2.2.6a) and (2.2.6b) to obtain
\[
\sum_{l=-\infty}^{\infty} \left\{ (-1)^{l+n} S_{l-n}^{H,G}(k, \kappa) [-(1 - \nu)kaJ_n'(ka) + ((1 - \nu)n^2 - k^2a^2) J_n(ka)] \\
+ \delta_l \left[ -(1 - \nu)kaH_n^{(1)'}(ka) + ((1 - \nu)n^2 - k^2a^2) H_n^{(1)'}(ka) \right] \right\} B_l \\
+ \sum_{l=-\infty}^{\infty} \left\{ (-1)^{l} S_{l-n}^{K,G}(k, \kappa) [-(1 - \nu)kaI_n'(ka) + ((1 - \nu)n^2 + k^2a^2) I_n(ka)] \\
+ \delta_l \left[ -(1 - \nu)kaK_n'(ka) + ((1 - \nu)n^2 + k^2a^2) K_n(ka) \right] \right\} D_l = \\
- \frac{\delta_0}{\sqrt{\chi_0}} i^n \left( \frac{\alpha_0 + i\chi_0}{k} \right)^n [-(1 - \nu)kaJ_n'(ka) + ((1 - \nu)n^2 - k^2a^2) J_n(ka)] \\
- \frac{\hat{\delta}_0}{\sqrt{\chi_0}} i^n \left( \frac{\alpha_0 + i\hat{\chi}_0}{k} \right)^n [-(1 - \nu)kaI_n'(ka) + ((1 - \nu)n^2 + k^2a^2) I_n(ka)], \tag{2.3.28a}
\]

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and

\[
\sum_{l=-\infty}^{\infty} \left\{ (-1)^l n G_{l-n}^H(k, \kappa) \left[ (1 - \nu)n^2 J_n(ka) - ((3 - \nu)n^2 ka + k^3 a^3) J'_n(ka) \right] + \delta_{ln} \left[ (1 - \nu)n^2 H_n^G(ka) - ((3 - \nu)n^2 ka + k^3 a^3) H'_n^G(ka) \right] \right\} B_l = \sum_{l=-\infty}^{\infty} \left\{ (-1)^l n G_{l-n}^K(k, \kappa) \left[ (1 - \nu)n^2 I_n(ka) + (1 - \nu)n^2 ka + k^3 a^3 \right] I'_n(ka) \right\} D_l = -\frac{\delta_0}{\sqrt{|\chi_0|}} \left( \frac{\alpha_0 + i\chi_0}{k} \right)^n \left[ (1 - \nu)n^2 J_n(ka) - ((3 - \nu)n^2 ka + k^3 a^3) J'_n(ka) \right], \quad \text{(2.3.28b)}
\]

after substitution of the dynamic Rayleigh identities. This can also be expressed in the matrix form (2.3.26) and solved as before.

Consequently from the systems (2.3.25a) and (2.3.25b) or (2.3.28a) and (2.3.28b) we can construct the solution \( w(x) \) within the strip \(-d/2 < x < d/2\) and form the solution throughout the entire field via the quasiperiodicity condition (2.3.13).

### 2.4 Bloch–Floquet waves in a doubly periodic array

In this section we continue with circular scatterers and provide a solution outline for a doubly periodic square array of cylinders. The solution procedure here is outlined in Chin et al. [25] and in Movchan et al. [105], and we begin with the derivation of the associated quasiperiodic Green’s functions.

#### 2.4.1 Quasiperiodic Green’s functions

For the two-dimensional lattice problem we now have quasiperiodic Green’s functions which satisfy the expressions

\[
(\Delta + k^2)G^H_a(x, x') = -\sum_p \delta(x - x' - \mathbf{R}_p)e^{ik \cdot \mathbf{R}_p}, \quad \text{(2.4.1a)}
\]

\[
(\Delta - k^2)G^M_a(x, x') = -\sum_p \delta(x - x' - \mathbf{R}_p)e^{ik \cdot \mathbf{R}_p}, \quad \text{(2.4.1b)}
\]

where for a square array of period \( d \) we have the array vector \( \mathbf{R}_p = (md, nd) \) for \( m, n \in \mathbb{Z} \), and \( \kappa \) which represents an arbitrary Bloch vector inside the first Brillouin zone. These satisfy the
quasiperiodicity conditions

\[ G^{H,M}_a(x + R_p, x') = G^{H,M}_a(x, x') e^{i \kappa R_p}, \]  
\[ G^{H,M}_a(x, x' + R_p) = G^{H,M}_a(x, x') e^{-i \kappa R_p}, \]

and we use the subscript \( a \) to distinguish the Green’s function here from the grating problem.

Subsequently one can write the associated Green’s functions as

\[ G^H_a(x, x') = \frac{i}{4} \sum_p H_0^{(1)}(k|\xi - R_p|) e^{i \kappa R_p}, \]  
\[ G^M_a(x, x') = \frac{i}{4} \sum_p H_0^{(1)}(ik|\xi - R_p|) e^{i \kappa R_p}, \]

where \( \xi = x - x' \) as before. Applying Graf’s addition theorem (2.3.4) we can obtain

\[ G^H_a(x, x') = \frac{i}{4} Y_0(k \xi) + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} S^{H,A}_l J_l(k \xi) e^{-il\gamma}, \]
\[ G^M_a(x, x') = \frac{1}{2\pi} K_0(k \xi) + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} S^{K,A}_l I_l(k \xi) e^{-il\gamma}, \]

where we have the array sums

\[ S^{H,A}_l(k, \kappa) = \sum_{p \neq (0,0)} H^{(1)}_l(k R_p) e^{i \phi_p} e^{i \kappa R_p}, \]
\[ S^{K,A}_l(k, \kappa) = \sum_{p \neq (0,0)} K_l(k R_p) e^{i \phi_p} e^{i \kappa R_p}, \]

with \( \gamma = \arg \xi \), and \( \phi_p = \arg R_p \), which converges provided that \( \xi < R_p \) (and is satisfied throughout the fundamental cell).

It is possible to simplify the array sum expression using the decomposition \( S^{H,A}_l(k, \kappa) = S^{J,A}_l(k, \kappa) + i S^{Y,A}_l(k, \kappa) \) where

\[ S^{J,A}_l(k, \kappa) = \sum_{p \neq (0,0)} J_l(k R_p) e^{i \phi_p} e^{i \kappa R_p}, \]
\[ S^{Y,A}_l(k, \kappa) = \sum_{p \neq (0,0)} Y_l(k R_p) e^{i \phi_p} e^{i \kappa R_p}, \]

and consequently the quasiperiodic Green’s function (2.4.4a) can take the form

\[ G^H_a(x, x') = \frac{1}{4} Y_0(k \xi) + \frac{1}{4} \sum_{l=-\infty}^{\infty} S^{Y,A}_l J_l(k \xi) e^{-il\gamma}. \]
2.4 Bloch–Floquet waves in a doubly periodic array

The efficient evaluation of the array sums (2.4.5b) and (2.4.5a) are considered at a later stage in Appendix B.

2.4.2 Derivation of the Rayleigh identities for a doubly periodic array

Using the quasiperiodic Green’s functions obtained in the preceding subsection, it is then possible to obtain Rayleigh identities for the doubly periodic problem as outlined in Movchan et al. [105]. This is identical to the grating case shown previously in section 2.3.2.

The only difference in the derivation is the use of the quasiperiodic Green’s function for the array

\[
G_a^H(x, x') = \frac{1}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr) J_n(kr') e^{in(\theta - \phi)}
\]

\[
+ \frac{1}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} S_{m-n}^{HA} J_n(kr) J_m(kr') e^{in\theta + im\phi}, \quad (2.4.8a)
\]

with the associated normal derivative

\[
\partial_n G_a^H(x, x') = \partial_r G_a^H(x, x') = \frac{k}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr) J_n'(kr') e^{in(\theta - \phi)}
\]

\[
+ \frac{k}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} S_{m-n}^{HA} J_n(kr) J_m'(kr') e^{in\theta + im\phi}, \quad (2.4.8b)
\]

where \( \theta = \arg r \) and \( \phi = \arg r' \). Ultimately we can determine that

\[
A_n = \sum_{m=-\infty}^{\infty} (-1)^{m+n} S_{m-n}^{HA} B_m, \quad (2.4.9)
\]

which is the desired Rayleigh identity for the Helmholtz component of the array problem. An identical treatment for the modified Helmholtz component of the solution yields the result

\[
C_m = \sum_{m=-\infty}^{\infty} (-1)^m S_{m-n}^{KA} D_m. \quad (2.4.10)
\]

2.4.3 Multipole solution for a doubly periodic array

Using the dynamic Rayleigh identities (2.4.9) and (2.4.10), the multipole expansion of the solution (2.1.4) and applying clamped-edge boundary conditions (2.2.3a) and (2.2.3) in an analogous
2.4 Bloch–Floquet waves in a doubly periodic array

manner to the grating problem we can obtain the system

$$\sum_{m=-\infty}^{\infty} \left[ (-1)^{n+m} S_{m-n}^{HA} J_n(ka) + H_n^{(1)}(ka) \delta_{mn} \right] B_m$$
$$+ \sum_{m=-\infty}^{\infty} \left[ (-1)^m S_{m-n}^{KA} I_n(ka) + K_n(ka) \delta_{mn} \right] D_m = 0, \quad (2.4.11a)$$

and

$$\sum_{m=-\infty}^{\infty} \left[ (-1)^{n+m} S_{m-n}^{HA} J'_n(ka) + H_n^{(1)'}(ka) \delta_{mn} \right] B_m$$
$$+ \sum_{m=-\infty}^{\infty} \left[ (-1)^m S_{m-n}^{KA} I'_n(ka) + K'_n(ka) \delta_{mn} \right] D_m = 0. \quad (2.4.11b)$$

Similarly one can impose free-edge boundary conditions (2.2.6a) and (2.2.6b) to obtain

$$\sum_{l=-\infty}^{\infty} \left\{ (-1)^{l-n} S_{l-n}^{HA}(k, \kappa) \left[ -(1-\nu)kaJ'_n(ka) + \left( (1-\nu)n^2 - k^2a^2 \right) J_n(ka) \right] \right. \right.$$\n
$$+ \delta_{ln} \left[ -(1-\nu)kaH_n^{(1)'}(ka) + \left( (1-\nu)n^2 - k^2a^2 \right) H_n^{(1)}(ka) \right] \} B_l$$
$$\sum_{l=-\infty}^{\infty} \left\{ (-1)^{l-n} S_{l-n}^{KA}(k, \kappa) \left[ -(1-\nu)kaI'_n(ka) + \left( (1-\nu)n^2 + k^2a^2 \right) I_n(ka) \right] \right. \right.$$\n
$$+ \delta_{ln} \left[ -(1-\nu)kaK'_n(ka) + \left( (1-\nu)n^2 + k^2a^2 \right) K_n(ka) \right] \} D_l = 0, \quad (2.4.12a)$$

and

$$\sum_{l=-\infty}^{\infty} \left\{ (-1)^{l-n} S_{l-n}^{HA}(k, \kappa) \left[ (1-\nu)n^2 J_n(ka) - \left( (3-\nu)n^2 ka + k^3a^3 \right) J'_n(ka) \right] \right. \right.$$\n
$$+ \delta_{ln} \left[ (1-\nu)n^2 H_n^{(3)}(ka) - \left( (3-\nu)n^2 ka + k^3a^3 \right) H_n^{(1)'}(ka) \right] \} B_l$$
$$\sum_{l=-\infty}^{\infty} \left\{ (-1)^{l-n} S_{l-n}^{KA}(k, \kappa) \left[ (1-\nu)n^2 I_n(ka) + \left( -(1-\nu)n^2 ka + k^3a^3 \right) I'_n(ka) \right] \right. \right.$$\n
$$+ \delta_{ln} \left[ (1-\nu)n^2 K_n(ka) + \left( -(1-\nu)n^2 ka + k^3a^3 \right) K_n(ka) \right] \} D_l = 0. \quad (2.4.12b)$$

These systems can be suitably truncated to form an associated block matrix system \( \textbf{M}(k, \kappa) \textbf{z} = \textbf{0} \). The associated band surfaces can then be determined by specifying Bloch vector coordinates \( \kappa \) which are typically contained within the first Brillouin Zone, and searching through \( k \) for vanishing determinant.

Note that this procedure can encounter difficulty when the associated matrix has a high condition number, or if \( k \) lies near a singularity of \( S_{l-n}^{YA}(k, \kappa) \), revealing false roots. Poulton et al. [124] propose that for every \( k \)-value such that \( \det(\textbf{M}(k, \kappa)) = 0 \), one should take a singular value
decomposition of the matrix $M$. Using the singular values $\sigma_i$ they compute $F_i = \sigma_{\text{min}}/\sigma_i$, where $\sigma_{\text{min}} = \min(\sigma_i)$. If this ratio $F_i > 1/100$ for any $i$ then the root is considered false.

2.5 Discussion

In this chapter we have provided an outline for the solution to several problems involving circular inclusions, which rely on the derivation of Rayleigh identities for the one- and two-dimensional array problems. Numerical solutions to these problems in turn depend on convergent expressions for grating and array sums which are outlined in the appendices of this thesis. The results and expressions in this chapter are useful for the pinned plate problem, which we consider now.
3.1 Introduction

In Chapter 2 we examined PlaCs consisting of circular inclusions of finite radius. In this chapter we outline the solution to the problem of a pinned elastic plate, which is equivalent to the case of a clamped circular cavity but with vanishing radius (i.e., $a \to 0$). In particular, we consider the problems of wave scattering by a single pin, a finite number of pins, a one-dimensional array of pins, and a doubly periodic array of pins. The theory outlined is given in Evans and Meylan [35], Evans and Porter [38], Kouzov and Lukyanov [69] and Movchan et al. [107] and is outlined here for completeness.
3.2 Single pinned point

We begin with the biharmonic plate equation in the frequency domain

\[(\Delta^2 - k^4) w(x) = (\Delta^2 + k^2) (\Delta^2 - k^2) w(x) = 0. \tag{3.2.1}\]

The associated Green’s function is given in closed form by

\[G^P(x, x') = \frac{i}{8k^2} \left[ H_0^{(1)}(k\xi) - \frac{2}{\pi i} K_0(k\xi) \right], \tag{3.2.2}\]

where \(\xi = x - x'\), with associated norm \(\xi\). Note that the Green’s function is regular for \(\xi \equiv 0\) and is given by \(G^P(x', x') = i/8k^2\).

We begin by considering an incident plane wave of the form

\[w_i^H(x) = \frac{\delta_0}{\sqrt{\vert \chi_0 \vert}} e^{i(\alpha_0 x - \chi_0 y)}, \tag{3.2.3}\]

where \(\alpha_0 = k \sin \theta_i\), \(\chi_0 = \sqrt{k^2 - \alpha_0^2} = k \cos \theta_i\) and \(\delta_0\) is the amplitude. It is then possible to decompose the solution into an incident and scattered field, where we write the scattered field in terms of the Green’s function for the biharmonic plate equation

\[w(x) = w_i^H(x) + a_0 G^P(x, 0), \tag{3.2.4}\]

and impose the condition of zero displacement at the origin

\[w(0) = 0. \tag{3.2.5}\]

The ansatz (3.2.4) follows the established solution procedure for pinned plates, as outlined in Evans and Porter [38], Norris and Vemula [111]. In Norris and Vemula [111], it was shown that as the radius of a clamped circular inclusion goes to zero, the second boundary condition \(\partial_n w |_{r=a} = 0\) vanishes, and the scattered field component of the displacement is proportional to the free-space Green’s function for the plate.

From this one can straightforwardly determine that \(a_0 = \frac{8k^2 i \delta_0}{\sqrt{\vert \chi_0 \vert}}\) and that the total field can be expressed as

\[w(x) = e^{i(\alpha_0 x - \chi_0 y)} + \frac{\delta_0}{\sqrt{\vert \chi_0 \vert}} \left[ \frac{2}{\pi i} K_0(kr) - H_0^{(1)}(kr) \right], \tag{3.2.6}\]

where \(r\) is defined as the norm of the vector \(x = (x, y)\).
3.3 Clusters of pins

For the case of multiple pins we can write the scattered field as a superposition of Green's functions centred about each pin, and consequently express the total field in the form

\[ w(x) = w^H_i(x) + \sum_{n=1}^{N} a_n G^P(x, x_n), \quad (3.3.1) \]

where \( a_n \) represents the amplitude corresponding to the \( n^{th} \) pin. At each pin location \( x_n \) we impose the condition

\[ w(x_n) = 0, \quad \text{for } n = 1, \ldots, N, \quad (3.3.2) \]

which reveals an \( N \times N \) linear system given by

\[ 0 = w^H_i(x_m) + \sum_{n=1}^{N} a_n G^P(x_m, x_n), \quad \text{for } m = 1, 2, \ldots, N, \quad (3.3.3) \]

which can be represented in matrix form and solved directly to recover the unknown \( a_n \) amplitudes, and thus determine the displacement at any point in the domain using (3.3.1).

3.4 An infinite grating of pinned points

For the case of a single array of pinned points subject to a plane incident wave, we can expand the total displacement as

\[ w(x) = w^H_i(x) + \sum_{n=-\infty}^{\infty} a_n G^P(x, x_n), \quad (3.4.1) \]

where \( x_n = (nd, 0) \) for \( n \in \mathbb{Z} \). As discussed in Chapter 2 we can impose the quasiperiodicity condition \( w(x + md, y) = w(x) \exp(i\alpha_0 md) \) to obtain the relation

\[ a_n = a_0 e^{i\alpha_0 nd}, \quad (3.4.2) \]

and since (3.4.2) accounts for the periodicity of the grating, we need only apply the boundary condition at the central pin. In other words, we impose the condition

\[ w(0) = 0, \]

revealing

\[ 1 + a_0 \sum_{n=-\infty}^{\infty} e^{i\alpha_0 nd} \left[ H^1_0(|k|n|d) - \frac{2}{\pi} K_0(|k|n|d) \right] = 0, \quad (3.4.3a) \]
or in the more compact form

\[ 1 + a_0 \left[ Z^H_G(k, \alpha_0, d) - \frac{2}{\pi i} S^K_G(k, \alpha_0, d) \right] = 0. \] (3.4.3b)

Here \( S^H_G \) and \( S^K_G \) are the familiar grating sums (2.3.6a) and (2.3.6b). Using the associated convergent representations these can be evaluated numerically and allow one to obtain

\[ a_0 = \frac{\pi i}{2 S^K_G(k, \alpha_0, d) - \pi i S^H_G(k, \alpha_0, d)}, \] (3.4.4)

providing the denominator is nonvanishing (i.e., away from a resonance of the structure). Using the amplitude \( a_0 \) we can compute the solution \( w(x) \) in the infinite strip \(-d/2 < x < d/2\), and recover the solution at any point in the domain using the quasiperiodicity condition given earlier.

### 3.5 Doubly periodic array of pinned points

We consider the case of a square array of pinned points (as well as finite clusters of pins) in the publication “Negative refraction and dispersion phenomena in platonic clusters”. We apply a number of concepts from Chapter 2 of the thesis, which are relevant to the solution for circular scatterers of finite radius \( a \), and apply these here. This work was largely inspired by the existing work on pinned plates by Evans and Porter [38], McPhedran et al. [99], Movchan et al. [106, 107]. That is, in Evans and Porter [38] the problem of computing the solution for a pinned elastic plate was given, however using the theory from McPhedran et al. [99], Movchan et al. [106, 107] we were able to investigate doubly periodic square arrays as well as Gaussian beam scattering by finite clusters of pinned points. The paper is largely self contained, and differs in notation to the main body of this thesis by the following points:

i. The variables \( k_0x, \ k_{px} \) and \( k_0y \) are used in place of \( \alpha_0, \alpha_p \) and \( \chi_0 \), respectively.

ii. In the paper we consider Helmholtz incident waves of unit amplitude (i.e. \( \delta_0 = \sqrt{|\chi_0|} \)), and denote these by \( w^i \) in place of \( w^H_i \).

iii. The variable \( r \) is used to denote \( \xi \) here.

iv. In the paper the sums \( S^H_m, S^Y_m \) and \( S^K_m \) denote the array sums \( S^H_A, S^Y_A \) and \( S^K_A \), respectively.

### 3.6 Publication
Negative refraction and dispersion phenomena in platonic clusters

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Abstract: We consider the problem of Gaussian beam scattering by finite arrays of pinned points, or platonic clusters, in a thin elastic plate governed by the biharmonic plate equation. Integral representations for Gaussian incident beams are constructed and numerically evaluated to demonstrate the different behaviours exhibited by these finite arrays. We show that it is possible to extend the scattering theory from infinite arrays of pinned points to these finite crystals, which exhibit the predicted behaviour well. Analytical expressions for the photonic superprism parameters $p$, $q$ and $r$, which are measures for dispersion inside the crystal, are also derived for the pinned plate problem here. We demonstrate the existence of negative refraction, beam splitting, Rayleigh anomalies, internal reflection, and near-trapping on the first band surface, giving examples for each of these behaviours.

3.6.1 Introduction

The study of platonic crystals (PlaCs), is an emerging field and a sub-branch of the study of photonic crystals. A platonic crystal is, by analogy to a photonic crystal, a regular arrangement of scatterers where the propagation medium is governed by the biharmonic thin plate equation as opposed to the Helmholtz equation. While the problem of scattering in thin plates has a considerable history \cite{42, 66, 111}, the problem is quite challenging even for a single cavity, and in fact, the solution for an arbitrary shaped hole has only been found recently \cite{135}. We are not interested here with complex scatterers but instead with the simplest scattering problem that can be imposed on a thin plate: scattering by regularly-spaced pins. This problem was first studied by Evans and Porter \cite{38} and has recently been the subject of extensive work by \cite{6, 53, 104, 106, 107}. It turns out that this problem, while relatively simple to compute numerically, displays a wide range of complex behaviours, ranging from near trapping \cite{104}, to an analogue to electromagnetically induced transparency effects \cite{53}. While the problem of scattering by pins is mainly of theoretical interest, the thin plate has recently been the subject of experiments by Stenger et al. \cite{145}, who, motivated by Farhat et al. \cite{42}, demonstrated strong cloaking for a singular circular scatterer by elastodynamic waves in a wide frequency range. The

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importance of this experiment is discussed in the review article by McPhedran and Movchan [95]. More complicated platonic crystals, such as those containing defects, have also been examined by Poulton et al. [125], Smith et al. [137]. One prominent feature of such crystals is the presence of modes which are highly localised within the defects.

For a PlaC comprised of pins (isotropic point scatterers), the linear system of equations can be formed and subsequently solved by direct inversion (with the exception of resonant frequencies, where the matrix is non-invertible [104]). This is a consequence of the fact that the Green’s function for the biharmonic plate equation is regular as \( kr \to 0 \), in contrast to the Green’s function for the Helmholtz equation, which is singular in the same limit. As such, the corresponding system for a cluster of pins under Helmholtz’s equation requires a slight modification in order to be solved analytically – a procedure known as Foldy’s method [44, 79]. Other techniques, such as homogenisation, could also be applied to compute the solution for such crystals, as shown by Krokhin et al. [71] for photonic crystals, and Poulton et al. [122] for periodic elastodynamic structures.

To understand the scattering behaviour of PlaCs a close examination of the band surfaces for the doubly periodic pinned problem is required. By using the band surfaces and the conservation condition at the interface one can obtain the group velocity vector direction, which determines the primary direction of the wave inside the periodic array of pins [57]. This method has been used to successfully predict the behaviour of water waves, photonic crystals and phononic crystals (i.e. [55, 116, 163]). However, this theory has never been applied to scattering in thin plates and we apply this approach for the first time here.

For a plate fixed at a doubly periodic set of points, an elegant dispersion relation is obtained in terms of elementary lattice sums, which was first shown by Movchan et al. [107], drawing upon an extensive body of research dedicated to the efficient evaluation of these sums in the context of photonic crystals, such as [25, 105, 106, 123, 153]. This dispersion relation reveals an infinite number of continuous band surfaces where propagation through the medium is supported. It is on the first band surface that we investigate the different scattering behaviours possible, including negative refraction. Band surfaces in photonic crystals have been used to explain several exotic refractive effects, including negative refraction, in which a wave is refracted at a photonic crystal interface by an angle with opposite sign to that of a homogeneous material. We discuss the conditions for negative refraction and associated dispersion phenomena for vibrations in thin plates for the first time here also.

In the context of constructing these slowness surfaces, the incident and refracted angle can exhibit extremely complex relations. For this reason Baba and Nakamura [8], Kosaka et al. [68], Steel et al. [144] investigated the parameters \( p, q \) and \( r \), which denote the generalised angular resolving power, generalised dispersion and resolution of the crystal respectively. These parameters examine the relationship between the frequency, incident angle and refraction angle of waves travelling through the structure, and thus act as measures of dispersion inside the
crystal. We determine these parameters for our problem here and derive a new expression for $q$ and $r$ in the platonic case. These parameters also help us identify where superprism or superlensing effects are possible in the platonic setting [116, 117].

The refraction of waves by these crystals is extremely complicated and is not well illustrated if excited by plane long crested waves. We use here Gaussian incident beams of the form given in Elsherbeni et al. [34], Marshall et al. [86] to clearly demonstrate the behaviour both inside and beyond the PlaC. These localised incident beams are constructed as a continuous superposition of travelling plane wave solutions to the Helmholtz equation. We demonstrate that the Gaussian beam can be efficiently evaluated by expanding it as a discrete superposition of plane waves using Gauss–Hermite quadrature. There is a large body of research dedicated to the construction of Gaussian wave packets in the time domain, as well as on the stability of Gaussian beams, and we direct the interested reader to [23, 24, 58, 61].

We show that negative refraction is observed inside the PlaC, and is typically characterised by a negative Goos-Hanchen shift on the exit interface of the crystal, for non-shallow angles. Negative refraction effects have been previously observed in metamaterial slabs [165], thin plates with square cavities [42] and photonic crystals [30, 114]. Here we also demonstrate that beam splitting [13] is possible for shallower angles of refraction. This is closely connected with the concept of shear wave splitting in seismology and geophysics [129], however we are unaware of any publications that identify beam splitting in a platonic context.

The outline of this paper is as follows. In Section 3.6.2 we introduce the solution to the problem of plane wave scattering by a finite PlaC. In Section 3.6.3 we outline how to construct Gaussian incident beams, and compute the response of the PlaC to these special incident waves. In Section 3.6.4 we consider the doubly periodic pin problem and compute the band surfaces to reveal information about the Bloch states of the crystal. In Section 3.6.5 we derive analytic expressions for the resolution parameters $p$, $q$ and $r$ which characterise both frequency and angular dispersion in platonic crystals. In Section 3.6.6 we show how to determine the angle of the reflected and transmitted beams. This is followed in Section 3.6.7 by a survey of the Brillouin zone, where a sample of behaviours is shown along with the $p$, $q$, $r$ and effective index surfaces for the first band surface (which is also given). In Section 3.6.8 we provide a brief summary of the paper followed by Appendix 3.6.9 which outlines the numerical approach for computing the Gaussian beams discussed in Section 3.6.3.

3.6.2 Plane wave incidence

The governing equation for a thin elastic plate in two dimensions is given by the Euler–Bernoulli equation

\[
(D \Delta^2 + \rho h \partial_t^2) u(x; t) = 0,
\]  
(P.1a)
3.6 Publication: “Negative refraction and dispersion phenomena in platonic clusters”

where \( u \) represents the out-of-plane displacement of the plate, \( \Delta \) is the Laplacian and \( D, \rho \) and \( h \) represent the material properties of the plate [152]. Assuming the solution is time-harmonic \( u(x; t) = \text{Re}\{w(x)e^{-i\omega t}\} \) we can obtain the frequency domain form of the plate equation

\[
(\Delta^2 - k^4)w(x) = (\Delta + k^2)(\Delta - k^2)w(x) = 0,
\]

(P.1b)

where \( k^2 = \omega \sqrt{\rho h/D} \) is the non-dimensionalised wave number and \( \omega \) denotes the angular frequency. We first consider the problem of wave scattering when plane waves are incident upon a finite square geometry of pins, where at each pin location \( x_n \) we impose the condition

\[
w(x_n) = 0, \quad \text{for } n = 1, \ldots, N.
\]

(P.2)

The solution method for scattering of a plane incident wave by a finite cluster of pins is well known [38, 107] and can be applied to any finite arrangement of pins, including irregular and defective geometries. It is included here for completeness. The incident wave is taken to be of the form

\[
w^i(x) = e^{i(\alpha_0 x - \chi_0 y)},
\]

(P.3)

where \( \alpha_0 = k \sin \theta_i \) and \( \chi_0 = \sqrt{k^2 - \alpha_0^2} = k \cos \theta_i \). A schematic of the problem is given in Figure 3.1, showing the orientation of the incident and refracted wave angles. We can consider each of these pins as representing point sources of various strengths. This allows us to expand the solution as the incident wave plus a superposition of Green’s functions centred about each pin:

\[
w(x) = w^i(x) + \sum_{n=1}^{N} a_n g(|x - x_n|),
\]

(P.4)

where \( a_n \) represents the amplitude corresponding to the \( n^{th} \) pin. The Green’s function \( g(r) \) satisfies

\[
(\Delta^2 - k^4)g(r) = \frac{\delta(r)}{2\pi r},
\]

(P.5a)

and is given in closed form by

\[
g(r) = \frac{i}{8k^2} \left[H_0^{(1)}(kr) - \frac{2}{\pi i} K_0(kr)\right],
\]

(P.5b)

where \( r = |x - x'| \), \( H_0^{(1)} \) is a zero-order Hankel function of the first kind and \( K_0 \) is a modified Bessel function of order zero [105, 123].

To solve this problem, we apply the boundary condition (P.2) at each of these pinned points admitting

\[
0 = w^i(x_m) + \sum_{n=1}^{N} a_n g(|x_m - x_n|), \quad \text{for } m = 1, 2, \ldots, N,
\]

(P.6)

which is an \( N \times N \) linear system that can be solved directly to recover the unknown \( a_n \) amplitudes, and thus determine the displacement at any point in the domain. For the platonic problem here we observe that the Green’s function is regular as \( r \) tends to zero \( (g(0) = i/8k^2) \),
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Figure 3.1: Infinite thin plate with a PlaC showing incident angle $\theta_i$ and angle of the group velocity $\theta_c$ for a localised incident wave.

In contrast to the Green’s function for Helmholtz’s equation which is unbounded.

Using this plane wave solution approach we can construct more complicated incident waves, including Gaussian beams. This is the topic of the following section.

3.6.3 Gaussian beam incidence

As outlined in [13, 158], solutions to Helmholtz’s equation

$$(\Delta + k^2) w(x, y) = 0,$$

can be expressed in the form

$$w(x, y) = \int_{-\infty}^{\infty} A(\alpha) e^{i\alpha x + i\chi(\alpha)y} d\alpha,$$  \hspace{1cm} (P.7a)

where

$$A(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w(x, 0) e^{-i\alpha x} dx,$$  \hspace{1cm} (P.7b)

$$\chi(\alpha) = \begin{cases} \sqrt{k^2 - \alpha^2}, & \text{if } |\alpha| \leq k, \\ i\sqrt{\alpha^2 - k^2} & \text{if } |\alpha| > k. \end{cases}$$  \hspace{1cm} (P.7c)

and $w(x, 0)$ denotes an initial beam profile at $y = 0$. Here we consider a one-dimensional Gaussian initial profile given by

$$w(x, 0) = \exp(-x^2/W^2) \exp(i\alpha_0 x),$$  \hspace{1cm} (P.8)

where $\exp(i\alpha_0 x)$ is the modulation envelope, and $2W$ is the full width at half maximum of the Gaussian (FWHM). Substituting this initial profile into expressions (P.7b) and (P.7a) we can
obtain the Gaussian beam expression

\[ w(x, y) = \frac{W}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\alpha-a_0)^2W^2/4} e^{i\alpha x + i\chi(\alpha)y} d\alpha, \quad (P.9) \]

which represents a Gaussian beam travelling in the direction of increasing \( y \) where the profile \( w(x, 0) \) is perfectly reconstructed at \( y = 0 \). Observe that in (P.9) the Gaussian beam is expressed as a continuous weighted superposition of plane waves that have a fixed wave number \( k \), but are considered over multiple incident directions, with the primary direction defined by the carrier wave value \( a_0 \).

It is a straightforward extension from (P.9) to construct a Gaussian incident wave where the initial profile is refocussed at some point \((x_0, y_0)\) above the cluster and travelling in the direction of decreasing \( y \). This expression is given by

\[ w_G(x, y) = \frac{W}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\alpha-a_0)^2W^2/4} e^{i\alpha(x-x_0) - i\chi(\alpha)(y-y_0)} d\alpha, \quad (P.10) \]

and is analogous to the form given in [34, 86]. Note that the Gaussian integral forms (P.10) and (P.9) have one strong disadvantage in that some of the component plane waves become evanescent at large angles of incidence (that is \( k^2 - \alpha^2 < 0 \)) causing a reduction in the plane wave component of the beam (but ultimately not contributing to the total wave field in the far-field region). To overcome this issue completely we can instead consider a normally incident Gaussian beam and rotate our coordinate system by the desired incident angle, revealing

\[ w_G(x, y) = \int_{-\infty}^{\infty} f(\alpha) e^{i\rho(\alpha)x - i\chi(\rho(\alpha))y} d\alpha, \quad (P.11) \]

where \( \rho(\alpha) = \alpha \cos \theta_i + \chi(\alpha) \sin \theta_i \) and \( f(\alpha) = W/(2\sqrt{\pi}) \exp\{-(\alpha W)^2/4\} \). Qualitatively, both of these expressions reveal identical scattering profiles, however the latter expression avoids the issue of component incident waves being evanescent completely and so we consider the final form (P.11) here.

Having constructed a Gaussian incident beam in the absence of a finite cluster we now need to include the scattering effects of our PlaC. Returning to Section 3.6.2 we recall that the scattered field for an incident plane wave is simply a sum of weighted Green’s functions centred about all the pinned points. Thus, the total field for a Gaussian incident beam, including the response, can be represented as

\[ w(x, y) = \int_{-\infty}^{\infty} f(\alpha) \left[ w_i(x) + \sum_{j=1}^{N} a_j g(|x - x_j|) \right] d\alpha, \quad (P.12) \]

where \( w_i(x) = \exp\{i\rho(\alpha)x - i\chi(\rho(\alpha))y\} \) and \( f(\alpha) \) represents the amplitude of each plane wave component.
3.6 Publication: “Negative refraction and dispersion phenomena in platonic clusters” \cite{57}

One final consideration in constructing Gaussian beams is to determine the size of \( W \). We can regard all of the Gaussian integral forms above as corresponding to an infinitely long screen containing a single aperture of width \( W \) with a source placed on one side. Provided that \( W \geq \lambda \), plane waves are able to pass through the aperture without interference, allowing the integral forms of the Gaussian to be constructed straightforwardly (which is a necessary condition for the representations given here). However if the width of the Gaussian \( W \) is small relative to the wavelength (\( W < \lambda \)) then circular waves emanate from the aperture creating difficulty in forming our required Gaussian. Finally, if \( W \) is large relative to the wavelength then plane waves emanate from the aperture, which can be easily shown by considering the definition of the Dirac delta function

\[
\lim_{\alpha \to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} = \lim_{\xi \to \infty} \frac{\xi}{2\sqrt{\pi}} e^{-\xi^2/4} = \delta(x),
\]

(P.13a)

which when substituted into (P.9) reveals

\[
\lim_{W \to \infty} w = \int_{-\infty}^{\infty} \delta(\alpha - \alpha_0) e^{i\alpha x + i\chi(\alpha)y} d\alpha = e^{i\alpha_0 x + i\chi(\alpha_0)y}.
\]

(P.13b)

We find that \( W \simeq 2\lambda \) works well in general for the expressions above.

3.6.4 Wave propagation in infinite periodic arrays

We now consider the simple problem of zero forcing (no incident wave) in a doubly periodic infinite array of pins, which are arranged in a square lattice of period \( d \). We represent the position of these pins by

\[
R_p = (nd, md), \quad p = (n, m), \quad m, n \in \mathbb{Z},
\]

(P.14)

and express the plate displacement as

\[
w(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn} g(|x - R_p|),
\]

(P.15)

where \( g(r) \) is the Green’s function given in (P.5b). Due to the perfect symmetry present in this problem, we need only consider computing the solution inside a fundamental cell around the origin known as the first Brillouin zone \cite{57}. Using the solution inside the Brillouin zone it is then possible to construct the solution throughout the entire plate via the quasiperiodicity condition

\[
w(x + R_p) = w(x) e^{i\kappa \cdot R_p},
\]

(P.16)

where \( \kappa = (\kappa_x, \kappa_y) \) denotes the Bloch wave vector inside the crystal. Using this quasiperiodicity condition, (P.15) can be expressed in the form

\[
w(x, y) = b_{00} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{i\kappa \cdot R_p} g(|x - R_p|).
\]

(P.17)
Since the quasiperiodicity has been built into the problem through the above condition, we need only apply the pin condition at the origin \( u(0,0) = 0 \), revealing

\[
\sum_p e^{-i\kappa \cdot R_p} g(|R_p|) = 0,
\]

(P.18)
after reversing the order of summation. The expression (P.18) can be simplified by use of Graf’s addition theorem which is given by formula (9.1.79) in Abramowitz and Stegun [1] as

\[
C_\nu(w)e^{i\nu\chi} = \sum_{l=-\infty}^{\infty} C_{l+\nu}(u)J_l(v)e^{il\alpha}, \quad \text{where} \quad |ve^{i\alpha}| < |u|,
\]

(P.19)
where the lines \( u, v \) and \( w \) form a closed triangle with \( w = \sqrt{u^2 + v^2 - 2uv \cos \alpha} \), \( \alpha \) denoting the angle between the lines \( u \) and \( v \), and \( \chi \) denoting the angle between the lines \( u \) and \( w \). Also, \( C_\nu \) denotes a general Bessel function \((J_\nu, Y_\nu, I_\nu \text{ or } K_\nu)\). Applying the pin condition \( u(\xi d, \eta d) = 0 \) gives

\[
\sum_p e^{-i\kappa \cdot R_p} g(|R_{p'} - R_p|) = 0,
\]

(P.20)
where \( R_{p'} = (\xi d, \eta d) \). Applying Graf’s addition theorem to each Bessel function element of the Green’s function in (P.20) reveals

\[
H_0^{(1)}(k|R_{p'} - R_p|) = \sum_{l=-\infty}^{\infty} H_l^{(1)}(k|R_p|)J_l(k|R_{p'}|)e^{il(\arg R_p - \arg R_{p'})},
\]

(P.21a)
\[
K_0(k|R_{p'} - R_p|) = \sum_{l=-\infty}^{\infty} K_l(k|R_p|)J_l(k|R_{p'}|)e^{il(\arg R_p - \arg R_{p'})},
\]

(P.21b)
which can only be applied when \( |R_p| > |R_{p'}| \). Substituting these two representations into (P.20) then allows us to obtain

\[
\left[ H_0^{(1)}(k|R_{p'}|) - 2\frac{\pi}{i}K_0(k|R_{p'}|) \right] + \sum_{l=-\infty}^{\infty} \left\{ S_l^H(k, \kappa) - 2\frac{\pi}{i}S_l^K(k, \kappa) \right\} J_l(k|R_{p'}|)e^{-il\arg R_{p'}} = 0,
\]

(P.22)
where we know that \( S_m^H(k, \kappa) = -\delta_{m,0} + iS_m^Y(k, \kappa) \),

\[
S_m^Y(k, \kappa) = \sum_{p \neq (0,0)} Y_m(k|R_p|)e^{im\arg R_p}e^{i\kappa \cdot R_p},
\]

(P.23a)
\[
S_m^K(k, \kappa) = \sum_{p \neq (0,0)} K_m(k|R_p|)e^{im\arg R_p}e^{i\kappa \cdot R_p},
\]

(P.23b)
from [25, 106]. If we now designate \( \xi = \eta = 0 \), the final form of the dispersion relation can be obtained involving monopole terms alone:

\[
S_0^Y(k, \kappa) + 2\pi S_0^K(k, \kappa) = 0.
\]

(P.24)
The final difficulty that remains is that these lattice sums \( S^Y_m \) and \( S^K_m \) suffer from poor convergence. To remedy this issue, Movchan et al. [106] used the method of repeated integration to obtain convergent lattice sum expressions for the case of a square array

\[
S^Y_m(k, \kappa) = \frac{1}{J_{m+3}(k\xi)} \left(-Y_3(k\xi) + \frac{1}{\pi} \sum_{n=1}^{3} \frac{(3-n)!}{(n-1)!} \left( \frac{2}{k\xi} \right)^{3-2n+2} \delta_{m,0} \right.
\]
\[
- \frac{4}{d^2} \frac{1}{m} \sum_p \left( \frac{k}{Q_p} \right)^3 \frac{J_{m+3}(Q_p \xi) e^{im\theta_p}}{Q_p^2 - k^2} \right), \quad (P.25a)
\]

and

\[
S^K_m(k, \kappa) = \frac{1}{I_{m+3}(k\xi)} \left( \left[ K_3(k\xi) - \frac{8}{(k\xi)^3} + \frac{1}{k\xi} - \frac{1}{8} k\xi \right] \delta_{m,0} \right.
\]
\[
+ \frac{2\pi}{d^2} \frac{1}{m} \sum_p \left( \frac{k^3}{Q_p^3} \right) \frac{J_{m+3}(Q_p \xi) e^{im\theta_p}}{Q_p^2 + k^2} \right), \quad (P.25b)
\]

where \( \delta_{mn} \) is the Kronecker delta, \( \theta_p = \arg Q_p, Q_p = (\kappa_x + 2\pi n/d, \kappa_y + 2\pi m/d), Q_p = ||Q_p||_2 \), and the vector \( \xi \), with corresponding norm \( \xi \), represents an arbitrary vector positioned inside the first Brillouin zone. Here we have used an acceleration parameter of 3, which is within the recommended range specified in Movchan et al. [106]. Care must be taken as numerical instability can be encountered when this parameter is too large [25].

An alternative expression to (P.24) can be obtained for our problem which is directly convergent and avoids the need to evaluate lattice sums completely. This is given in [137]. It is from (P.24) that we are able to obtain band surfaces using the accelerated lattice sums defined above. That is, for a given Bloch vector \( \kappa \) the dispersion relation reveals values of \( k \) where propagation through the crystal is supported.

### 3.6.5 Platonic resolution parameters

Using the band surfaces, we know that it is a straightforward procedure to determine the direction of the group velocity inside a photonic crystal for a given incident angle and wave vector from inspection, assuming the crystal is suitably large [57]. However, the band diagram does not provide a full picture of how wave energy can behave inside a PlaC. To this end, Baba and Matsumoto [7] introduced three parameters: \( p, q, \) and \( r \), to examine the degree of beam collimation, the degree of wavelength sensitivity, and the resolution of an incident beam inside a photonic crystal. We extend the theory from photonic crystals to our platonic problem here.
based on the theory outlined in [7, 8, 144] and we begin with the definitions

\[
p = \frac{\partial \theta_c}{\partial \theta_i}, \quad (P.26a) \\
q = \frac{\partial \theta_c}{\partial \omega} |_{\theta_i}, \quad (P.26b) \\
r = \frac{q}{p} = -\frac{\partial \theta_i}{\partial \omega} |_{\theta_c}, \quad (P.26c)
\]

where \( \theta_c \) is the angle of refraction inside the crystal, as shown in Figure 3.1. To reiterate, \( p \) measures the sensitivity of \( \theta_c \) to changes in input angle (for fixed frequency), \( q \) measures the change in refraction angle with respect to frequency (for fixed input angle), and \( r \) is defined as the resolution parameter, as in [7]. This theory was developed for the design of high resolution superprisms which are highly sensitive to both changes of input frequency and incident angle, in order to steer the beam to desired angles. The theory outlined in this section is given in [144].

Here we examine PlaCs with flat entrance and exit interfaces only, and so there is a simplification of the model outlined. One of the core differences theoretically is that our dispersion relation is given by \( \omega = k^2 \), (that is, we specify \( \sqrt{\rho h/D} = 1 \) for all results here). This implies that outside the crystal \( \omega = \alpha_0^2 + \chi_0^2 \), and inside the crystal \( \omega = k^2 \) for \( k \) values satisfying equation (P.24). The incident wave angle and refraction angle are defined as

\[
\theta_c = \tan^{-1} \left( \frac{v_x}{v_y} \right), \quad \text{and} \quad \theta_i = \tan^{-1} \left( \frac{\alpha_0}{\chi_0} \right), \quad (P.27)
\]

where \( \mathbf{v} = (v_x, v_y) \) represents the group velocity vector inside the crystal, and the parameters \( \alpha_0 \) and \( \chi_0 \) are defined in (P.3).

Returning to the definition of \( p \) in (P.26a) we can see that (after omitting the \( |_\omega \) for ease of notation)

\[
p = \frac{\partial \theta_c}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \tan^{-1} \left( \frac{v_x}{v_y} \right) = \frac{1}{|\mathbf{v}|^2} \left( v_y \frac{\partial v_x}{\partial \theta_i} - v_x \frac{\partial v_y}{\partial \theta_i} \right), \quad (P.28a)
\]

which can be expressed in matrix form as

\[
p = \frac{1}{|\mathbf{v}|^2} \begin{bmatrix} v_y & -v_x \\ \vdots & \vdots \\ \frac{\partial v_x/\partial \theta_i}{\partial v_y/\partial \theta_i} & \end{bmatrix}. \quad (P.28b)
\]

In order to evaluate this expression we observe that the group velocity has the following dependence: \( \mathbf{v} = \mathbf{v}(\kappa \{ \mathbf{k}_0(\theta_i, \omega) \}) \), where \( \kappa = (\kappa_x, \kappa_y) \) is the PlaC wave vector and \( \mathbf{k}_0 = (\alpha_0, \chi_0) \) is defined as the incident wave vector. After applying the chain rule we can represent these group
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velocity derivatives in matrix form directly from the definition of $\theta_i$ in (P.27), that is

$$
\begin{bmatrix}
\frac{\partial v_x}{\partial \theta_i} \\
\frac{\partial v_y}{\partial \theta_i}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial^2 \omega}{\partial \kappa_x \partial v_x} & \frac{\partial^2 \omega}{\partial \kappa_y \partial v_y} \\
\frac{\partial^2 \omega}{\partial \kappa_x \partial \kappa_x} & \frac{\partial^2 \omega}{\partial \kappa_y \partial \kappa_y}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \kappa_x}{\partial \alpha_0} & \frac{\partial \kappa_y}{\partial \chi_0} \\
\frac{\partial \kappa_x}{\partial \kappa_x} & \frac{\partial \kappa_y}{\partial \kappa_y}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \alpha_0}{\partial \theta_i} \\
\frac{\partial \chi_0}{\partial \theta_i}
\end{bmatrix},
$$

(P.29a)

where

$$
\begin{bmatrix}
\frac{\partial \kappa_x}{\partial \alpha_0} \\
\frac{\partial \kappa_y}{\partial \alpha_0} \\
\frac{\partial \kappa_x}{\partial \chi_0} \\
\frac{\partial \kappa_y}{\partial \chi_0}
\end{bmatrix} = \left[\begin{array}{cc}
\frac{\partial \alpha_0}{\partial \kappa_x} & \frac{\partial \alpha_0}{\partial \kappa_y} \\
\frac{\partial \chi_0}{\partial \kappa_x} & \frac{\partial \chi_0}{\partial \kappa_y}
\end{array}\right]^{-1},
$$

(P.29b)

by virtue of inverse function theorem. To construct the second matrix of (P.29b) we make use of the conservation requirement that the tangential component of the incident and PlaC wave vectors must match at the interface, i.e. $\alpha_0 = \kappa_x$ (which is a requirement in order for wave energy to be admitted into the crystal). Therefore $\frac{\partial \alpha_0}{\partial \kappa_x} = 1$ and $\frac{\partial \alpha_0}{\partial \kappa_y} = 0$. We also apply the $\partial/\partial \kappa_x$ and $\partial/\partial \kappa_y$ operators to the dispersion relation for the plate outside the crystal to reveal

$$
\frac{\partial \chi_0}{\partial \kappa_x} = 1 \\
\frac{\partial \alpha_0}{\partial \kappa_x} = 1 \\
\frac{\partial \omega}{\partial \kappa_x} = \frac{1}{2 \chi_0} \left( \frac{\partial \omega}{\partial \kappa_x} - 2 \alpha_0 \right) \\
\frac{\partial \omega}{\partial \kappa_y} = \frac{1}{2 \chi_0} \frac{\partial \omega}{\partial \kappa_y}
$$

(P.30)

thus allowing us to construct the first unknown matrix

$$
\begin{bmatrix}
\frac{\partial \kappa_x}{\partial \alpha_0} & \frac{\partial \kappa_x}{\partial \chi_0} \\
\frac{\partial \kappa_y}{\partial \alpha_0} & \frac{\partial \kappa_y}{\partial \chi_0}
\end{bmatrix} = \left[\begin{array}{cc}
1 & 0 \\
\frac{1}{2 \chi_0} \left( \frac{\partial \omega}{\partial \kappa_x} - 2 \alpha_0 \right) & \frac{1}{2 \chi_0} \frac{\partial \omega}{\partial \kappa_y}
\end{array}\right]^{-1} = \left[\begin{array}{cc}
1 & 0 \\
\frac{2 \alpha_0 - \partial \omega/\partial \kappa_x}{\partial \omega/\partial \kappa_x} & \frac{2 \chi_0}{\partial \omega/\partial \kappa_y}
\end{array}\right].
$$

(P.31a)

To determine the final unknown vector in (P.29a) we can make use of the inverse function theorem once more and construct a Jacobian matrix involving derivatives of the incident wave vector $k_0$ with respect to $\theta_i$ and $\omega$:

$$
\begin{bmatrix}
\frac{\partial \alpha_0}{\partial \theta_i} & \frac{\partial \alpha_0}{\partial \omega} \\
\frac{\partial \chi_0}{\partial \theta_i} & \frac{\partial \chi_0}{\partial \omega}
\end{bmatrix} = \left[\begin{array}{cc}
\frac{\partial \theta_i}{\partial \alpha_0} & \frac{\partial \theta_i}{\partial \chi_0} \\
\frac{\partial \omega}{\partial \alpha_0} & \frac{\partial \omega}{\partial \chi_0}
\end{array}\right]^{-1}.
$$

(P.31b)

Note that our desired vector is given by the first column of (P.31b). We can differentiate the dispersion relation for the crystal outside the plate directly to reveal $\partial \omega/\partial \alpha_0 = 2 \alpha_0$ and $\partial \omega/\partial \chi_0 = 2 \chi_0$, and from the definition of $\theta_i$ in (P.27) we can compute the remaining elements

$$
\frac{\partial \theta_i}{\partial \alpha_0} = \frac{\chi_0}{\omega} \\
\frac{\partial \theta_i}{\partial \chi_0} = -\frac{\alpha_0}{\omega}.
$$

(P.32)
For the purposes of calculating $q$ we return to the original definition given in (P.26b) (after omitting the $|\theta_i$ for ease of notation) to obtain

$$q = \frac{\partial \theta_c}{\partial \omega} \tan^{-1} \left( \frac{v_x}{v_y} \right) = \frac{1}{|v|^2} \begin{bmatrix} v_x & -v_y \end{bmatrix} \frac{\partial v_x/\partial \omega}{\partial v_y/\partial \omega}.$$ 

(P.35)

where the second matrix is given by (P.31a) and the final vector is given by the second column of (P.33). The resulting substitutions allow us to obtain the generalised dispersion

$$q = \frac{1}{|v|^2} \begin{bmatrix} \frac{\partial \omega}{\partial \kappa_y} & \frac{\partial \omega}{\partial \kappa_x} \end{bmatrix} \begin{bmatrix} \frac{\partial ^2 \omega}{\partial \kappa_x \partial \kappa_x} & \frac{\partial ^2 \omega}{\partial \kappa_x \partial \kappa_y} \\ \frac{\partial ^2 \omega}{\partial \kappa_y \partial \kappa_x} & \frac{\partial ^2 \omega}{\partial \kappa_y \partial \kappa_y} \end{bmatrix} \begin{bmatrix} \frac{\partial \kappa_x}{\partial \omega} & \frac{\partial \kappa_x}{\partial \kappa_y} \\ \frac{\partial \kappa_y}{\partial \omega} & \frac{\partial \kappa_y}{\partial \kappa_y} \end{bmatrix} \begin{bmatrix} \frac{\partial v_x}{\partial \omega} & \frac{\partial v_y}{\partial \omega} \end{bmatrix},$$

(P.36)

which is slightly different from the photonic $q$ expression in [144] due to a $1/2$ scaling factor in the first term. This arises from the fact that the dispersion relation for a PlaC is of the form $\omega = k^2$ as opposed to $\omega = k$ for photonic crystals. From the definition of $p$ and $q$ outlined above,
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it is a straightforward task to directly evaluate the platonic resolution expression

\[ r = \frac{1}{\chi_0} \left( \frac{\chi_0}{2\omega} + \frac{\chi_0}{|v|^2} \frac{\partial^2 \omega}{\partial \kappa_x \partial \kappa_y} \left( \partial \omega / \partial \kappa_y - \frac{\partial^2 \omega / \partial \kappa_y^2}{\partial \omega / \partial \kappa_y} \right) \right), \tag{P.38} \]

which also differs by a 1/2 scaling factor in the first term compared to the photonic case.

3.6.6 Outside the crystal

It is a well known fact that the tangential component of the incident wave vector is always conserved when entering any number of different media in the plate [57] (i.e., here \( \kappa_x = \alpha_0 \)). One direct consequence of this is that the grating equation completely determines all reflected and transmitted output angles above and below the crystal, regardless of crystal depth and group velocity.

The problem of computing output angles using the grating equation is a straightforward task; the angles are determined by

\[ k_{px} = k \sin \theta_p = \alpha_0 + 2\pi p / d \tag{P.39} \]

where \( p \in \mathbb{Z} \), [13]. We can determine which orders are propagating and which are evanescent by computing

\[ \chi_p = \begin{cases} \sqrt{k^2 - k_{px}^2}, & \text{if } |k_{px}| \leq k, \\ 1/\sqrt{k_{px}^2 - k^2}, & \text{if } |k_{px}| > k, \end{cases} \tag{P.40} \]

where real values of \( \chi_p \) are associated with propagating orders. Consequently from (P.39) we can determine the required \( \theta_p \) angles at which waves exit the crystal from below at \((x_{out}, y_{out}) = (-dc_d \tan \theta_c, -dc_d)\), where \( c_d \) is the depth of the crystal.

3.6.7 Results and discussion

We construct the Gaussian beam outlined in the first section of this paper and consider a beam half-width \( W = 5 \). The following images are constructed using a cluster of 61 pins across \((-30 \text{ to } 30)\) and 11 gratings deep \((0 \text{ to } -10)\) in total.

This simple system exhibits very complex behaviour; our aim here is to illustrate some interesting features of the first band surface, and is by no means exhaustive.

We begin by computing the equal value contours of \( k \) for the first band surface, from (P.24)
(Figure 3.2). For convenience we refer to these as isofrequency contours hereafter, even though these contours are shown as functions of wave number $k$ and not the corresponding frequency $\omega$. We observe a relative maximum at $(\kappa_x, \kappa_y) = (0, 0)$, absolute maxima at $(\pm \pi, \pm \pi)$, absolute minima at $(0, \pm \pi)$ and $(\pm \pi, 0)$, and saddle points at $(\pm 1.6650, \pm 1.6650)$. For any periodic structure it is usually the case that the absolute maxima and absolute minima of a band surface occur at the edges of the Brillouin zone \[19\]. The first band occupies the interval $\pi < k < 4.4434$, below which a complete stop band exists, where no propagation through the crystal is possible at all. The first band surface for the pinned plate problem exhibits a much more complicated behaviour compared to the band surfaces seen in Steel et al. \[144\] for a photonic crystal of small-radius air holes in silicon, and Farhat et al. \[42\] for an array of square holes in a thin plate.

Returning to Section 3.6.3, we recall that any incident Gaussian beams must satisfy $W \geq 2\pi/k$ and therefore a minimum waist of $W \geq 2$ must be considered for scattering on the first band surface.

Figure 3.3 shows the $p$, $q$ and $r$ contour plots for the platonic problem, where we follow the convention in Steel et al. \[144\] and plot $\log_{10}|1/p|$, $\log_{10}|q|$ and $\log_{10}|r|$, as $\log_{10}|r|$ is the sum of these other two quantities by definition. These contours are quite different from the photonic case shown in \[144\] due largely to the very different isofrequency contours in the platonic case, which exhibit a stronger degree of curvature comparatively. For our Gaussian beam scattering problem, which involves multiple incident directions $\theta_i$, at a fixed frequency $k$, the most useful parameter here is $p$. Had we constructed a Gaussian beam in the time domain, which involves a superposition of plane waves of different wave numbers, then $q$ (and subsequently $r$) would become a parameter of greater interest. Incidentally, we have been able to replicate these surfaces
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Figure 3.3: (a) Surface plot of $\log_{10} |1/p|$, (b) $\log_{10} |q|$, (c) $\log_{10} |r|$ and (d) $\log_{10} |n_1/n_2|$ over the entire Brillouin zone.

from the first principle definitions of these parameters, which gives us strong confidence in these results.

The final image 3.3(d) is the effective refractive index of the crystal, which follows directly from Snell’s law $n_1/n_2 = \sin(\theta_i)/\sin(\theta_c)$, where $n_{1,2}$ denotes the refractive index of the first and second media respectively. This parameter $n_1/n_2$ also gives the ratio of the phase velocities inside the PlaC, allowing us to determine the speed with which the energy is transmitted through the crystal (Born and Wolf [13]).

On all of these plots we see standard and predictable behaviour in the form of smooth surfaces, with the exception of fine lines corresponding to extreme values which dissect the Brillouin zone, in part due to the curvature of the band surface. Notably, we see four interwoven vertical arcs in all of these surfaces that correspond to extremely high and extremely low parameter values, which intersect at the saddle points on the first band surface. Gaussian beams sent in
at frequencies and angles which sit in the vicinity of this weaved structure are associated with near-trapped waves in the crystal which are highly leaky.

In Figure 3.3(a) we have two light parabolic arcs corresponding to small $|p|$, a line of high $|p|$ at $\kappa_y = 0$, and our four interwoven arcs of high and low $|p|$ which sublend the two parabolic arcs at the saddle points. These light parabolic arcs correspond to locations on the first band surface where the strongest vertical collimation is observed (and where the PlaC is least sensitive to changes in incident angle). The line at $\kappa_y = 0$ is where the surface is most sensitive to changes in incident angle. This line of high $|p|$ theoretically reveals where superlensing is possible (a superlens is a photonic crystal where $\theta_c$ varies rapidly with changes in $\theta_i$ as discussed in Farhat et al. [42], and in a different context by Pendry and Smith [117]).

In Figure 3.3(b) we have a vertical line of low sensitivity to changes in wavelength at $\kappa_x = 0$ and a horizontal line of high sensitivity along $\kappa_y = 0$. We also have small corner parabolas of low sensitivity in all four corners of the Brillouin zone in addition to our four interwoven arcs seen previously in the $\log_{10}|1/p|$ figure.

In Figure 3.3(c) we observe all of the features seen in the two preceding figures, except that the line $\kappa_y = 0$ is now undefined as it corresponds to $q = 0$.

The final Figure 3.3(d) is a surface of $\log_{10}|n_1/n_2|$, which shows that the speed of the wavefronts inside the crystal is at its lowest along the two horizontal parabolic arcs seen in Figure 3.3(a), reaching a minimum near the origin. There is a vertical line of infinite effective index corresponding to where $\theta_c$ is zero. Such very high values of refractive index correspond to what is termed ultra refraction in the photonic crystal literature [93]. Using this surface one could easily determine thin segments on the first band surface where the effective index is $n_1/n_2 = -1$, and thus obtain the perfect lens discussed by Pendry [116], however due to the finiteness of the PlaC this may be difficult to observe and is not examined here.

We now move onto examining some examples of the different behaviours seen inside the first band surface in Figures 3.4 to 3.10. For a number of the examples given (such as Figure 3.4(a)) we provide corresponding slowness contours, which are figures of the band surface contours at a desired $k$ value, superposed with the interface conservation condition $\alpha_0$ given by the broken red line, and a blue circle which represents all possible incident wave directions at our fixed $k$. An associated incident wave vector is shown at the desired $\theta_i$ angle, and the edge of the first Brillouin zone is outlined by a grey box. The origin of the group velocity vector is given by the intersection of the broken red line with the solid black contour cuts (because of the periodicity there are usually only two cuts, and we choose the cut which physically corresponds to the direction where energy is admitted into the crystal). The direction of the group velocity vector is given by the normal of the isofrequency contour at this intersection point, pointing in the direction of increasing $k$. 
Also shown are pictures of the displacement $\text{Re}(u)$ inside the PlaC, and a picture of the crystal and the surrounding plate region (such as Figures 3.4(b) and (c)). For the larger picture, the incident ray vectors and associated crystal vectors are superposed on top of the field unless they extend well beyond the $x$ limits of the figure. Note that as these are pictures of the displacement in the frequency domain, we view each of these field images as (usually) different snapshots in time of the time-harmonic displacement field.

We begin in Figure 3.4 with an example of negative refraction. In particular, Figure 3.4(a) is the slowness contour which clearly shows the direction of the group velocity at our chosen input parameters (given by the magenta arrow). At this particular $k$ we also see the location of partial stop bands, which corresponds to where the vertical $\alpha_0$ line would not intersect with a band surface contour. Figure 3.4(b) provides a close-up of the plate displacement inside the crystal, which clearly shows the wavefronts rotated at a negative angle relative to the incident beam angle. However, we note here that the angle of these phase fronts does not always correspond to the direction of wave energy propagation, which is always given by the group velocity angle. That is, the phase velocity and group velocity are usually quite disconnected inside the crystal, which demonstrates the highly anisotropic nature of PlaCs. This is reinforced in Figure 3.4(c) where the arrow within the platonic crystal structure indicates the direction of the group velocity, which we note does not coincide with the phase front direction.

Inside the finite grating stack we can see some leakage horizontally along the cluster along the direction of periodicity. This small variation in the behaviour is entirely to be expected given that we are applying infinite theory to PlaCs of finite size, and thus that we are not able to achieve perfectly regular behaviour within the crystal. Regardless of this, the theory is still able to work well in achieving the desired behaviour, as demonstrated by Figure 3.4(c). Here we see the Gaussian beam striking the cluster from above, with a weaker beam reflected above the cluster. We also see a strong transmitted beam which has undergone a negative Goos-Hanchen shift on the exit interface of the crystal, giving us confirmation that negative refraction has actually taken place inside the crystal [13]. The reflected and transmitted beams may take some time to resolve to their predicted angles, given the local evanescent behaviour of the PlaC.

In Figure 3.5 we provide a typical example of platonic beam splitting, which is observed in a large section of the first band surface. Beam splitting is associated with both high $|p|$ and effective index values as there is strong vertical collimation, $\theta_c \simeq 0$, implying that the phase velocity inside the crystal is slow compared to the incident beam. The slowness contour in Figure 3.5(a) predicts a very shallow angle of refraction inside the crystal, which is confirmed in the subsequent images of the displacement. Figure 3.5(b) also shows that the phase fronts inside are highly rotated relative to the group velocity angle, and Figure 3.5(c) shows the excitation of two beams above and below the PlaC. Notably, above the cluster the excited beam interacts with the incident Gaussian creating a clear trough of minimal displacement.

Since the angle of refraction for beam splitting is often negative, it technically doubles as an
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example of negative refraction, however, no negative Goos-Hanchen shift is clearly observable and in its place we have two output beams, which can be explained by (P.39) above as two orders are excited \( \chi_{0,-1} \in \mathbb{R} \) under such a configuration.

To demonstrate that perfect vertical collimation is possible under this setting we provide an example of Littrow splitting in Figure 3.6. This is a platonic analogue to the Littrow configuration seen in optics [13], where the grating orders are excited in such a way that the incident beam is reflected back from the direction at which it came (i.e., retroreflected), creating a perfectly symmetric field. We can determine when this takes place by setting \( \theta_i = -\theta_p \) in (P.39) to obtain the Littrow condition

\[
2k \sin(\theta_i) = 2\pi p/d.
\]

Compared to Figure 3.5 we can see that it only takes a small change in incident angle to generate this behaviour because of the curvature of the band surface. We can observe this most clearly by comparing the slowness contours in Figures 3.5(a) and 3.6(a). In this setting the direction
of the phase fronts also coincides with the direction of the group velocity. In Figure 3.7 we give surface plots of the plate displacement when Rayleigh anomalies are encountered. Since Rayleigh anomalies are not a PlaC effect, but purely a grating effect, the slowness contours are not given, but generally speaking, Rayleigh anomalies are characterised by a ‘passing off’ effect where additional orders are excited ($\chi_{\pm p}$ for $p \neq 0$ switches from a purely imaginary number to a small, real value) giving rise to a secondary output beam which appears almost parallel to the grating [39, 94, 96, 97, 147, 164]. The location of these anomalies can be determined analytically by specifying $\theta_i = \pm \pi/2$ in (P.39) and solving appropriately.

In Figure 3.7(a) we can clearly see this surface wave travelling in the direction of $x \rightarrow -\infty$ both above and below the crystal. Above the PlaC we can see a small trough of minimal displacement caused by the interaction of the Rayleigh anomaly wave with the incident Gaussian beam. For this particular example we have $\chi_{-1} = 0.0662$, $\chi_0 = 2.3180$ and $\text{Im}(\chi_p) > 8$ for all other $p$ demonstrating that only a very small angular change is required to observe this effect.

We show one further example of the different scattering behaviour exhibited by PlaCs for simi-
larly shallow θc values in Figure 3.8. Here we see the formation of clearly defined peaks inside the crystal, which appear to be formed by the Gaussian beam reflecting off the far wall of the PlaC. These antisymmetric peaks are strongest in even numbered channels, and clearly outline the zero displacement condition along each row of pins. Above the crystal we see a shallow reflected wave forming a clear channel of minimal displacement. This particular configuration corresponds to a high effective index (and |p| value), which is typically associated with high reflection in photonics.

The final two examples in Figures 3.9 and 3.10 correspond to situations where standard ray tracing methods fail completely, as a unique physical cut for the group velocity vector in the slowness contours cannot be obtained.

In particular, Figure 3.9 has an infinite number of directions supported where the α0 line intersects the band surface contours, and Figure 3.10(a) yields multiple directions (two rays are possible here). In both cases the behaviour inside the crystal is associated with near trapped waves where our matrix of Green’s functions has complex roots with |Im(k)| ≃ 0. These two
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Figure 3.7: (a), (b): Displacement (Re(u)) of the plate (where \( k = 3.5697, \alpha_0 = 2.7174, \theta_i = 49.57^\circ \)) demonstrating a Rayleigh anomaly.

examples here sit in the neighbourhood of the interwoven arcs of high and low \(|p|\), where a similar behaviour is observed throughout. These highly leaky waveguide modes can propagate some distance inside the crystal, but nevertheless cause a complicated pattern inside, as seen in Figures 3.9(b), 3.9(c), 3.10(b) and 3.10(c).

3.6.8 Summary

In this paper we have demonstrated how to construct the solution to the problem of Gaussian beam incidence on a PlaC. From the homogeneous problem of a doubly periodic infinite array of pins we were able to construct the first band surface for a square array, which was then used to determine the primary wave direction inside the PlaC. In addition to this we have computed the resolution parameters \( p, q, r \) and the effective index for our doubly periodic array to determine the input angle sensitivity and speed of the waves inside. We have given a brief survey of the diverse wave phenomena across the first Brillouin zone revealing how this simple platonic system exhibits many of the complex behaviours observed in photonic crystals. Here we do not observe any qualitative differences between the two crystal types, which is consistent with the fact that the biharmonic plate equation decomposes into Helmholtz (and modified Helmholtz) operators. However, quantitative differences could be expected for various other phenomena, since the presence of the modified Helmholtz operator can potentially strengthen the role of evanescent over propagating waves, leading to stronger trapped wave effects. Thus the photonic crystal literature can be used as a guide in the construction of novel devices for guiding and dispersing flexural waves in structured plates.
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Figure 3.8: (a), (b): Displacement ($\text{Re}(u)$) of the plate (where $k = 3.6373$, $\alpha_0 = 0.50$, $\theta_i = 7.90^\circ$) demonstrating internal reflection.

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3.6.9 Appendix: Numerical evaluation

To evaluate the infinite integral in (P.11) we use Gauss–Hermite quadrature

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \simeq \sum_{l=1}^{M} w_l f(x_l),$$

as outlined in (25.4.46) of [1], where $x_l$ is the $l^{th}$ zero of the Hermite polynomial $H_M(x)$,

$$w_l = \frac{2^{M-1} M! \sqrt{\pi}}{M^2 [H_{M-1}(x_l)]^2},$$

are the associated weights, and $M$ is the number of sampling points.

This allows us to numerically evaluate the finite integral representation as the following sum

$$w_{lG}(x, y) = \sum_{l=1}^{M} f_l e^{i\beta x - i\chi(\beta)y}$$
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Figure 3.9: (a): Slowness contour, and (b), (c): displacement (Re(u)) of the plate (where \( k = 3.6239, \alpha_0 = 1.665, \theta_i = 27.35^\circ \)) demonstrating behaviour at the saddle point (near trapped waves).

where

\[
 \beta_l = \rho (2\alpha_l/W), \quad \text{and} \quad \chi(\alpha) = \begin{cases} 
 \sqrt{k^2 - \alpha^2}, & \text{if } |\alpha| \leq k, \\
 i\sqrt{\alpha^2 - k^2} & \text{if } |\alpha| > k.
\end{cases}
\]  

(P.42b)

To include the effect of the crystal we apply an identical technique to the evaluation of (P.12) and sum over all the plane wave fields to obtain

\[
 w_G(x, y) = \sum_{l=1}^{M} \beta_l \left\{ e^{i\beta_l x - i\chi(\beta_l)y} + \sum_{n=1}^{N} a_n G(x, x_n) \right\}.
\]  

(P.43)

We observe that (P.42a) and (P.43) are both truncated Fourier series representations, and accordingly have a finite period which depends strongly on the sampling parameter \( M \).

Given that our PlaC is finite, we can force the periodically repeating Gaussian beams out of our
region of interest around the cluster by specifying a suitably large $M$. We find that $M \simeq 500$ works well, however we only compute the plane wave fields for $|f_l| > 10^{-7}$ and thus for most cases approximately 74 plane wave solutions are considered. This is of greater importance when considering Gaussian beam scattering by infinite grating layers as interactions between the beams may occur for complex scattering cases.
3.7 Discussion

In this chapter we have considered the design of several PlaC geometries which are comprised of pinned points. This chapter has also featured the paper “Negative refraction and dispersion phenomena in platonic clusters” where we considered Gaussian beam scattering by finite clusters of pins and demonstrated a number of interesting diffraction behaviours. We now consider the case when defects are present in one- and two-dimensional pinned arrays in the following chapter.
4 Defects in pinned elastic plates

4.1 Introduction

In Chapter 3 we considered uninterrupted arrays of pinned points in one and two dimensions. In this chapter we consider the implications of when pins are removed from such perfectly periodic structures. That is, for one-dimensional arrays we provide a solution to the problem of wave scattering by a defective grating, and for two-dimensional arrays we consider the problem of when several pins are removed (in any configuration), as well as the removal of multiple lines of pins. For both the one- and two-dimensional problems, the solution to the scattering problem is expressed in terms of the solution for the problem of a forced pin.

The paper “The effect on bending waves by defects in pinned elastic plates”, which constitutes this entire chapter, is an extension to previous work on pinned plates by Evans and Porter [38] as well as existing work done by the author on pinned PlaCs [136]. The solution method involves the use of Fourier transform methods for defects in one- and two-dimensional arrays comprised of pins and was derived independently from the work by Poulton et al. [125] who constructed the solution for defects in two-dimensional arrays of pins and point masses using Green’s function methods. The publication included here is self contained in terms of notation, but we scale the problem differently to previous chapters.
Instead of considering the biharmonic plate equation

\[(\Delta^2 - k^4) w(x, y) = 0,\]  \hspace{1cm} (P.1)

for a one-dimensional array of period \(d\), with vanishing displacement at discrete points along the \(x\)-axis

\[w(md, 0) = 0, \quad m \in \mathbb{Z},\]  \hspace{1cm} (P.2)

we introduce \(\bar{x} = kx, \bar{y} = ky, \bar{d} = kd\) and \(\bar{w} = kw\) and consider

\[(\bar{\Delta}^2 - 1) \bar{w}(\bar{x}, \bar{y}) = 0,\]  \hspace{1cm} (P.3)

where \(\bar{\Delta} = \partial^2_{\bar{x}} + \partial^2_{\bar{y}}\) with

\[\bar{w}(m\bar{d}, 0) = 0, \quad m \in \mathbb{Z},\]  \hspace{1cm} (P.4)

That is, we scale out the wave number \(k\) from the problem and investigate changes in the non-dimensionalised period \(\bar{d}\) of a single grating, and changes in the aspect ratio of two-dimensional rectangular arrays. Note that in the paper the bar and standard notations above are reversed to simplify the expressions (i.e., so that bar notation is not used throughout).

There are also the following changes in notation:

i. We use \(a\) to denote the period of a one-dimensional array, previously given by \(d\).

ii. The paper uses \(u\) to denote the time-harmonic displacement \(w\).

iii. In the paper we consider incident waves of unit amplitude (i.e. \(\delta_0 = \sqrt{|\chi_0|}\)).

iv. The Green’s function for the plate \(G^p\) is given by \(g(x, y)\) and we restrict the source point \(x'\) to the origin throughout.

v. The parameter \(\lambda(t)\) is used in place of \(\chi(t)\), and \((\rho, \chi)\) denotes a polar coordinate system.

vi. In the paper \(\phi\) denotes the Fourier space (spectral) parameter corresponding to \(y\).

vii. The paper examines flexural wave propagation through rectangular arrays by considering the aspect ratio \(b/a\), where \(a\) is the non-dimensionalised period in the \(x\) direction, as opposed to considering square arrays alone.

viii. We denote the group velocity \(v\) by \(c_g\).
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The effect on bending waves by defects in pinned elastic plates

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Abstract: This paper presents solutions to a number of problems posed for the out-of-plane displacement of infinite thin elastic plates that are rigidly pinned in periodic configurations, but that possess a finite number of ‘defects’. We begin by considering a single one-dimensional periodic array of pins. We derive an analytic solution for the displacement produced by the forced oscillation of the central pin in the array, and this solution is shown to be closely connected to the problem of scattering of plane waves by an array when a finite number of pins are removed. Attention then focuses on doubly-periodic rectangular arrays of pinned points possessing defects. Central to approaching such problems is an understanding of Bloch–Floquet waves in periodic arrays in the absence of defects and a simple method is described for computing the associated dispersion surfaces. The solution to three problems are then sought: the trapping of localised waves by a finite number of missing pins, trapping of waves by entire rows of missing pins, and the wave radiation pattern due to the forcing of a single pin. All problems are treated analytically by using bounded Green’s functions for thin elastic plates, a discrete Fourier transform solution method and simple, explicit and rapidly convergent evaluations of the one- and two-dimensional lattice sums that arise.

4.2.1 Introduction

Thin elastic plates are used in many engineering applications and are often either bonded to a substructure along ribs or rigidly pinned by rivets. Determining the transmission properties due to defects in the vibrations along periodically ribbed elastic sheets and membranes was the subject of a series of significant papers published over a number of decades, e.g. [28, 29, 75, 139, 143]. In these investigations the elastic membrane, or thin elastic plate, is viewed in cross-section and bounded above by a two-dimensional acoustic or fluid medium. The interest in such problems lies in how sound waves couple with vibrational modes on a periodically-supported elastic beam. The defects consisted of vibrating ribs and laterally displaced rib supports, the latter giving rise to localisation effects [140].

Evans and Porter [36, 38], motivated in part by the work mentioned above, considered so-called Very Large Floating Structures in which they imagined a large two-dimensional thin elastic sheet.
secured to the sea bed by mooring lines that provided periodic point supports on the elastic sheet. In this situation, although the underlying three-dimensional incompressible fluid does not support body waves, coupling still exists between the fluid and elastic sheet. Their work highlighted the mathematical elegance of using point sources (or Green’s functions) to represent the effect of point forces on an elastic sheet. In particular, whilst wave theories such as acoustics, electromagnetics and elasticity are governed by second-order partial differential equations, the Kirchhoff equation for a thin elastic plate is fourth order in space. Consequently, point sources behave like \( r^2 \log r \) as the distance \( r \) to the source is decreased rather than diverging like \( \log r \) as in source solutions to second order wave equations. This fact permits source functions to be used as physical representations of small clamped circular pins (see section 3.2 or Norris and Vemula [111] for reference). We remark that isotropic point sources can be used as an approximate model for acoustic wave scatterers in the limit of small, widely-spaced, soft-sound cylinders and long wavelengths, as discussed in section 8.2.5 of Martin [89].

A variety of work on two-dimensional thin elastic sheets has followed, mainly choosing to ignore the complication of coupling to an external fluid or acoustic medium (such problems have been examined by [4, 69, 70] for both arrays of pins and point masses). For example, Movchan et al. [106] computed the band-gap structure for a doubly periodic arrangement of holes of finite radius with either clamped or free edges. The dispersion relation for Bloch–Floquet waves through a periodically pinned sheet was obtained analytically by taking the limit of a clamped hole radius tending to zero. This particular problem, whose solution also appears in Evans and Porter [36], is revisited in section 4.2.5 of the current paper where a simpler approach to computing the band-gap structure is described. Other recent related works on pinned elastic plates include [53, 104, 107] and [115].

The focus of the present paper is to examine the effect of introducing ‘defects’ into both one- and two-dimensional infinite periodic arrays of rigidly pinned points in a thin elastic plate. In this paper a defect will mean either removing one or more pins from the array, or replacing a rigid pin by one which is forced to oscillate periodically in time at a prescribed amplitude. The mathematical difficulty in solving defect problems of this type arises as the geometry is no longer periodic.

Defects in one-dimensional periodic arrays have been considered in the setting of the two-dimensional Helmholtz equation for a linear array of acoustically-hard or soft cylinders by Thompson and Linton [149]. There the solution is approached using the so-called ‘Modified Array Scanning Method’. Its use of transform methods and excitation of waves by point sources to represent the defects bears similarities to the approach used in the present paper. However, our methods proceed more directly and transparently, evidently due to the simplicity afforded by using single point sources to represent pins. Earlier, Thompson and Linton [148] had used the array scanning method to consider the excitation of an acoustic wave field by a line source in the presence of a periodic array of cylinders.
Defects in doubly-periodic arrays are of current interest in several research fields including elastodynamics [131], phononic crystals [60] and photonic crystals [43, 85, 123, 128, 159]. Techniques for determining defect modes range from the method of fictitious sources, to supercell methods and multipole methods. Attention has also been focused on understanding the effects of defects in more complicated photonic structures such as woodpiles [59].

Recently, Poulton et al. [125] published related work considering the problems of single point and line defects which independently reproduces some of the results here. They also considered forcing and defects in doubly periodic arrays of point masses, as well as rigid pins. The interaction theory developed herein is able to accommodate multiple defects, as well as single defects, for doubly periodic arrays of rigid pins.

Bloch–Floquet problems occur frequently in many application areas of the physical sciences and central to the understanding of defects in doubly-periodic arrays is the consideration of the associated homogeneous problem without defects. Often they are related to the solution of the wave (or Helmholtz’s) equation which requires the evaluation of lattice sums and these in their most basic form are poorly convergent. Hence acceleration of lattice sums for computational purposes is crucial. As a result, efficient methods for the evaluation of convergent lattice sums (using Graf’s addition theorem and integral techniques) for Helmholtz’s equation in a doubly periodic domain continue to be developed, e.g. [25, 105, 121]. Linton [78] provides an exhaustive survey of the most commonly used techniques.

In contrast to those methods cited above for Helmholtz’s equation, here we are able to derive convergent, readily computable lattice sums for problems posed using the thin elastic plate equation using standard methods without the need to accelerate convergence characteristics. This is evidently a consequence of the low order of the condition applied at pinned points (zero) compared to the order of the governing equation (four). In particular, this means that the Green’s function that produces the point sources used to represent the pinned points is bounded everywhere.

Our general method of solution is applied first in section 4.2.2 to one-dimensional arrays. The roots of this method can be traced to Crighton [29] and are based on Fourier transforming the infinite systems of equations that arise from the application of pin conditions. Two distinct problems naturally arise: the wave radiation pattern due to the time-harmonic forcing of the central pin; and the scattering of plane waves by a number of missing pins in an array.

Solution methods for linear periodic arrays are then extended to more complicated and arguably more interesting problems involving doubly-periodic arrays of pins, and here we investigate the possible localised modes that are supported by defects. In total, three problems are considered here: the effect of removing one or more pins from the plate; the wave radiation pattern due to the forced motion of a single pin; and the effect of removing entire rows of pins. These are presented in section 4.2.5, and in section 4.2.9, and a selection of results are presented from each
of the problems considered in the paper. The work is summarised in section 4.2.14 where an indication is given as to how these methods may be extended to other related and physically interesting problems.

4.2.2 Defects in a single periodic array of pinned points

An infinite thin elastic plate occupies the \((\bar{x}, \bar{y})\) plane having an out-of-plane displacement \(\bar{u}(\bar{x}, \bar{y})\), and is assumed to be pinned rigidly along the line \(\bar{y} = 0\) at \(\bar{x} = m\bar{a}\) for \(m \in \mathbb{Z}\setminus \mathcal{M}\) where \(\mathcal{M}\) is a finite set which represents points in an otherwise periodic array which are not rigidly pinned (i.e. defects). In the simplest case \(\mathcal{M} = \{0\}\) represents a single defect at the origin. The period of the array is represented by \(\bar{a}\), and we consider two problems in this section. In the first, \(\mathcal{M} = \{0\}\) and in its place, the point \((0, 0)\) is excited by a forcing of amplitude \(C\) at angular frequency \(\omega\). (This problem shall be referred to as the forced pin problem.) In the second problem, a periodic array with defects is excited by an incident plane wave of fixed amplitude and angular frequency \(\omega\) from infinity, propagating at an angle \(\theta_0\) with the negative \(\bar{y}\)-axis.

In both cases, the governing equation for a thin elastic plate is given by

\[
(\bar{\Delta}^2 - k^4)\bar{u}(\bar{x}, \bar{y}) = 0,
\]  
(Q.1)

after removing a time dependence of \(e^{-i\omega t}\). Here \(\bar{\Delta}\) is the two-dimensional Laplacian, \(k^4 = m\omega^2/D\), \(m = \rho h\) is the mass per unit area in terms of the plate thickness \(h\) and density \(\rho\), and \(D = (1/12)Eh^3/(1 - \nu^2)\) is the flexural rigidity defined in terms of Young’s modulus \(E\) and Poisson’s ratio \(\nu\). The conditions at the fixed pins are

\[
\bar{u}(m\bar{a}, 0) = 0, \quad m \in \mathbb{Z}\setminus \mathcal{M}.
\]  
(Q.2)

Non-dimensionalising lengths with respect to \(k\) via \(x = k\bar{x}, \ y = k\bar{y}\) and \(a = k\bar{a}\) with \(u(x, y) = k\bar{u}(\bar{x}, \bar{y})\) converts (Q.1) and (Q.2) into

\[
(\Delta^2 - 1)u(x, y) = 0,
\]  
(Q.3)

and

\[
u(ma, 0) = 0, \quad m \in \mathbb{Z}\setminus \mathcal{M}.
\]  
(Q.4)

Thus the problem depends only on the single dimensionless parameter \(a\).

We first consider the problem of pin forcing at the centre of the one-dimensional periodic array. This solution is then used to construct the solution to the scattering of plane waves by a defective array in the following subsection.
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4.2.3 Forcing of the central pin

Here we consider the problem of forcing a single pin with unit amplitude in a periodic linear array of pinned points in the absence of an incident wave field. We use the superscript \((f)\) throughout to distinguish this problem from later problems, and introduce the notation \(u_m^{(f)} = u_m^{(f)}(ma, 0), m \in \mathbb{Z}\), to represent the displacement at the point \(x = ma\) in the array. For this problem \(\mathcal{M} = \{0\}\) and the pinned conditions \(u_m^{(f)} = 0\) are set for \(m \notin \mathcal{M}\) whilst at the origin \(u_0^{(f)} = 1\) is imposed. The total displacement of the plate can be written as

\[
u^{(f)}(x, y) = \sum_{n=-\infty}^{\infty} a_n^{(f)} g(x - na, y), \quad (Q.5)
\]

where \(g(x, y)\) represents a Green’s function for a source placed at the origin of a thin plate satisfying

\[
(\Delta^2 - 1)g(x, y) = \delta(x)\delta(y), \quad (Q.6)
\]

and is given explicitly by

\[
g(x, y) = \frac{i}{8} (H_0(r) - H_0(ir)), \quad (Q.7)
\]

where \(H_0 \equiv H_0^{(1)}\) represents a Hankel function of the first kind and \(r^2 = x^2 + y^2\). Note that at the origin the Green’s function is bounded as \(g(0, 0) = i/8\).

In (Q.5), the coefficients \(a_n^{(f)}\) are to be determined. Enforcing the boundary conditions at each point in the array on (Q.5) gives

\[
u_m^{(f)} = \sum_{n=-\infty}^{\infty} a_n^{(f)} g((m - n)a, 0) = \delta_{m0}, \quad (Q.8)
\]

for all \(m\), where \(\delta_{mn}\) represents the Kronecker delta function. Accordingly, multiplying through by \(e^{-im\theta}\) and summing over all \(m\) results in

\[
1 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_n^{(f)} g((m - n)a, 0)e^{-im\theta}, \quad (Q.9)
\]

where \(\theta\) refers to the standard angular polar coordinate.

We now define the following finite Fourier transforms (Fourier series) with

\[
A_n^{(f)}(\theta) = \sum_{n=-\infty}^{\infty} a_n^{(f)} e^{-in\theta}, \quad a_n^{(f)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_n^{(f)}(\theta) e^{in\theta} \, d\theta, \quad (Q.10a)
\]

\[
G_0(\theta; a) = \sum_{n=-\infty}^{\infty} g(na, 0)e^{-in\theta}, \quad g(na, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_0(\theta; a) e^{in\theta} \, d\theta. \quad (Q.10b)
\]

The series in (Q.10b) has been computed in Evans and Porter [38] and it helps to outline this.

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process here. Thus the integral representations of the Hankel functions allow us to write the Green’s function as

\[ g(x, y) = \frac{1}{8\pi} \int_{-\infty}^{\infty} \mathcal{G}(t, y)e^{xt}dt, \quad \mathcal{G}(t, y) = \frac{e^{-\lambda(t)|y|}}{\lambda(t)} - \frac{e^{-\gamma(t)|y|}}{\gamma(t)}, \quad (Q.11) \]

where

\[ \lambda(t) = \begin{cases} (t^2 - 1)^{1/2}, & t \geq 1, \\ -i(1 - t^2)^{1/2}, & -1 < t < 1, \\ (t^2 - 1)^{1/2}, & t \leq -1, \end{cases} \]

and

\[ \gamma(t) = (1 + t^2)^{1/2}. \quad (Q.12) \]

Note that \( \lambda(t) \) is decomposed in this manner to emphasize that the integrand \( \mathcal{G} \) is analytic on an appropriately cut Riemann surface. Alternative integral representations involving paths of integration in the complex plane can be found in Watson [157]. Applying the Poisson summation formula [38]

\[ 2\pi \sum_{n=-\infty}^{\infty} f(2n\pi) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\pm inu} f(u) du, \quad (Q.13) \]

to expression (Q.11) at the points \((x, y) = (na, 0)\) and comparing with (Q.10b) readily admits the convergent series

\[ G_0(\theta; a) = \frac{1}{4a} \sum_{n=-\infty}^{\infty} \mathcal{G}(t_n, 0), \quad (Q.14) \]

where \( t_n = (\theta + 2n\pi)/a \). It can be shown that the summand \( \mathcal{G}(t_n, 0) \) has a leading order asymptotic behaviour of \( a^3/(2\pi|n|^3) \) as \(|n| \to \infty\), so convergence can be accelerated by writing

\[ G_0(\theta; a) = \frac{1}{4a} \left( \mathcal{G}(t_0, 0) + \frac{2a^3}{(2\pi)^3} \zeta(3) \right) + \frac{1}{4a} \sum_{n=1}^{\infty} \left( \mathcal{G}(t_n, 0) + \mathcal{G}(t_{-n}, 0) - \frac{2a^3}{(2n\pi)^3} \right), \quad (Q.15) \]

where \( \zeta \) denotes the Riemann zeta function. This forces (Q.15) to converge like \( O(|n|^{-5}) \). It can also be seen that \( G_0 \) is both symmetric and periodic:

\[ G_0(-\theta; a) = G_0(\theta; a), \quad G_0(\theta + 2m\pi; a) = G_0(\theta; a) \quad \text{for } m \in \mathbb{Z}. \quad (Q.16) \]

The definitions above allow expression (Q.9) for the forcing of a single pin to be written in the form

\[ 1 = A^{(f)}(\theta)G_0(\theta; a), \quad (Q.17) \]

after using the convolution result for Fourier series. Rearranging (Q.17) for \( A^{(f)}(\theta) \) and inverting from (Q.10a) admits

\[ a_n^{(f)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_0(\theta; a) e^{in\theta} d\theta, \quad (Q.18) \]

which is the final form of the scattering coefficient.

It is proved in Evans and Porter [38] that \( G_0(\theta; a) \) does not vanish, and this postpones the
complications associated with singularities in integrals that we encounter in later parts of the paper.

As a corollary to the above solution, we can see from (Q.8) that

\[
\sum_{n=-\infty}^{\infty} a_{n-r}^{(f)} g((m-n)a,0) = \sum_{n'=-\infty}^{\infty} a_{n'}^{(f)} g((m-r-n')a,0) = \delta_{m-r,0} = \delta_{mr}. \tag{Q.19}
\]

Therefore, once the coefficients \(a_{n}^{(f)}\) have been determined from the problem of forcing a pin at the origin, the coefficients needed for forcing the \(r\)th pin are just \(a_{n-r}^{(f)}\).

To determine the far-field behaviour of the displacement, we can decompose the integral representation of the Green’s function (Q.11) into oscillatory and exponentially decaying integrals in terms of \(y\) to obtain:

\[
g(x,y) = \frac{i}{8\pi} \int_{-1}^{1} \frac{e^{ixt+i(1-t^2)^{1/2}|y|}}{(1-t^2)^{1/2}} dt + \frac{1}{8\pi} \int_{-\infty}^{\infty} \mathcal{G}(t,y)e^{ixt} dt, \tag{Q.20a}
\]

where

\[
\mathcal{G}(t,y) = \begin{cases} 
\frac{-e^{-\gamma(t)|y|}}{(t^2-1)^{1/2}} - \frac{e^{-\gamma(t)|y|}}{\gamma(t)}, & \text{if } |t| \geq 1.
\end{cases} \tag{Q.20b}
\]

Substituting (Q.20a) into (Q.5), we note that the dominant contribution to the far-field displacement will come from the first integral in (Q.20a) as it decays like \(\rho^{-1/2}\) (where \(x+iy = \rho e^{i\chi}\) in polar coordinates) due to its phase becoming stationary, while the second integral involving \(\mathcal{G}\) is \(O(\rho^{-3/2})\). Therefore after using the Fourier series definition (Q.10a), we find that

\[
u^{(f)} = \frac{1}{8\pi} \int_{-\infty}^{\infty} A^{(f)}(at)\mathcal{G}(t,y)e^{ixt} dt \\
\sim \frac{i}{8\pi} \int_{-1}^{1} \frac{A^{(f)}(at)}{(1-t^2)^{1/2}} e^{ixt+i(1-t^2)^{1/2}|y|} dt, \quad \text{as } \rho \to \infty. \tag{Q.21}
\]

Now, due to the symmetry of our problem, we can restrict ourselves to the half-plane \(y > 0\), and only consider \(0 < \chi < \pi\). Then (Q.21) can be rewritten as

\[
u^{(f)} \sim \frac{i}{8\pi} \int_{0}^{\pi} A^{(f)}(a \cos \tau)e^{i\rho \cos(\tau-\chi)} d\tau, \quad \text{as } \rho \to \infty. \tag{Q.22}
\]

Applying the method of stationary phase, one can directly obtain the result

\[
u^{(f)} \sim \left(\frac{2}{\pi \rho}\right)^{1/2} e^{i(\rho-\pi/4)} A_{\infty}^{(f)}(\chi), \quad \text{as } \rho \to \infty, \tag{Q.23a}
\]
where the diffraction amplitude is

\[ A_{\infty}^{(f)}(\chi) = \frac{i}{8} A^{(f)}(a \cos \chi) \]  

(\text{Q.23b})

This result can also be obtained by substituting the asymptotic form of the Green’s function for large arguments from (\text{Q.7}) into (\text{Q.5}), and can be continued into the half-plane \(-\pi < \chi < 0\) by reflection in the array. The asymptotics of similar integrals to (\text{Q.20a}) are considered by Norris and Wang [112], using a slightly different method.

### 4.2.4 Scattering of plane waves by defects in a 1D periodic array

We now move on to considering plane wave scattering by a linear periodic array of pinned points containing a finite number of missing pins, encoded in the set \(\mathcal{M}\). The superscript \((s)\) is used to denote quantities associated with this scattering problem. Here, we let \(u_{n}^{(s)} = u^{(s)}(ma, 0)\) for all \(m\) and so \(u_{m}^{(s)} = 0\) for \(m \in \mathbb{Z}\backslash \mathcal{M}\). The total displacement is written as

\[ u^{(s)}(x, y) = u^{(i)}(x, y) + \sum_{n=-\infty}^{\infty} a_{n}^{(s)} g((m-n)a, y), \]  

(\text{Q.24})

where the prescribed incident wave plate displacement is given by

\[ u^{(i)}(x, y) = e^{i\alpha_{0}x + \lambda(\alpha_{0})y}, \]  

(\text{Q.25})

with \(\alpha_{0} = \sin \theta_{0}\) and \(\lambda(\alpha_{0}) = -i \cos \theta_{0}\) as defined in (\text{Q.12}).

In expression (\text{Q.24}), the coefficients \(a_{n}^{(s)}, n \notin \mathcal{M}\) are to be determined, whilst we set \(a_{n}^{(s)} = 0\) for \(n \in \mathcal{M}\) since there is no contribution to the scattered field from pins that are removed from the array. Enforcing the pinned conditions on the general solution (\text{Q.24}) gives

\[ 0 = u^{(i)}(ma, 0) + \sum_{n=-\infty}^{\infty} a_{n}^{(s)} g((m-n)a, 0), \quad m \notin \mathcal{M}. \]  

(\text{Q.26})

For \(m \in \mathcal{M}\), the left-hand side of the above expression is replaced with the unknown displacement at each of the removed pins,

\[ u_{m}^{(s)} = u^{(i)}(ma, 0) + \sum_{n=-\infty}^{\infty} a_{n}^{(s)} g((m-n)a, 0), \quad m \in \mathcal{M}. \]  

(\text{Q.27})

Equations (\text{Q.26}) and (\text{Q.27}) can consequently be combined and written in the suggestive form

\[ \sum_{r \in \mathcal{M}} u_{r}^{(s)} \delta_{mr} = u^{(i)}(ma, 0) + \sum_{n=-\infty}^{\infty} a_{n}^{(s)} g((m-n)a, 0), \quad m \in \mathbb{Z}. \]  

(\text{Q.28})

The structure of (\text{Q.28}) allows a solution to be written as a superposition of the separate effects
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of incident wave scattering by an unbroken periodic array and forcing of strength \( u_r^{(s)} \) at each of locations \( r \in \mathcal{M} \) in the absence of an incident wave. In other words,

\[
a_n^{(s)} = a_n^{(u)} + \sum_{r \in \mathcal{M}} u_r^{(s)} a_{n-r}^{(f)},
\]

(Q.29)

which follows from the use of the forcing solution (Q.19). Here the coefficients \( a_n^{(f)} \) have been previously determined by (Q.18), whilst \( a_n^{(u)} \) is the solution of

\[
-a^{(i)}(ma,0) \equiv -e^{ina_0a} = \sum_{n=-\infty}^{\infty} a_n^{(u)} g((m - n)a,0), \quad m \in \mathbb{Z},
\]

(Q.30)

which is the equation for plane wave scattering by an uninterrupted grating. The periodicity of the left-hand side of (Q.30) implies a periodicity of the solution, i.e., \( a_n^{(u)} = a_0^{(u)} e^{ina_0a} \). Substituting this in (Q.30) gives

\[
-1 = a_0^{(u)} \sum_{n=-\infty}^{\infty} g(-na,0)e^{ina_0a} = a_0^{(u)} G_0(-a_0a;a) = a_0^{(u)} G_0(a_0a;a),
\]

after using (Q.10b) and (Q.16), and so the final form of (Q.29) is

\[
a_n^{(s)} = -\frac{e^{ina_0a}}{G_0(a_0a;a)} + \sum_{r \in \mathcal{M}} u_r^{(s)} a_{n-r}^{(f)}.
\]

(Q.32)

The remaining unknowns, \( u_r^{(s)} \) for \( r \in \mathcal{M} \), are determined by imposing the remaining condition \( a_n^{(s)} = 0 \) for \( n \in \mathcal{M} \) in (Q.32) resulting in the linear system

\[
\frac{e^{ina_0a}}{G_0(a_0a;a)} = \sum_{r \in \mathcal{M}} u_r^{(s)} a_{n-r}^{(f)}, \quad n \in \mathcal{M}.
\]

(Q.33)

When a single pin at the origin is missing, that is \( \mathcal{M} = \{0\} \), the solution of (Q.32) and (Q.33) is given explicitly by

\[
a_n^{(s)} = \frac{1}{G_0(a_0a;a)\left(-a_0^{(f)} G_0(a_0a;a)\right)} \left(-e^{ina_0a} + \frac{a_n^{(f)}}{a_0^{(f)}}\right).
\]

(Q.34)

In the scattering problem, there are two components to the far-field: plane waves reflected by an uninterrupted periodic grating and circular waves emanating from the defects. In the case of a single missing pin at the origin, these circular waves are easily identified from the second term in (Q.34) to be related to those for the forcing problem, so that the diffraction coefficient for the circular wave component of the scattered field is simply

\[
A_{\infty}^{(s)}(\chi) = \frac{A_0^{(f)}(\chi)}{a_0^{(f)} G_0(a_0a;a)},
\]

(Q.35)

with \( A_0^{(f)} \) defined by (Q.23b). More generally, for multiple missing pins, the contribution from
the sum in (Q.32) to far-field circular waves results in a diffraction coefficient given by

$$A_{\infty}^{(s)}(\chi) = A_{\infty}^{(f)}(\chi) \sum_{m \in \mathcal{M}} u_{m}^{(s)} e^{-ima\cos \chi}. \quad (Q.36)$$

The first term in the right-hand side of either (Q.32) or (Q.34) accounts for the diffracted wave field from an unbroken periodic array and its contribution to the total displacement may be written as

$$u^{(a)}(x, y) = -\frac{1}{G_0(\alpha_0 a; a)} \sum_{n=-\infty}^{\infty} e^{ina} g(x - na, y). \quad (Q.37)$$

Using the integral representation (Q.11) in the above and invoking Poisson’s summation formula gives

$$u^{(a)}(x, y) = -\frac{1}{4aG_0(\alpha_0 a; a)} \sum_{n=-\infty}^{\infty} e^{ina} \left( \frac{e^{-\lambda(\alpha_n)|y|}}{\lambda(\alpha_n)} - \frac{e^{-\gamma(\alpha_n)|y|}}{\gamma(\alpha_n)} \right), \quad (Q.38)$$

where

$$\alpha_n = \alpha_0 + 2n\pi/a. \quad (Q.39)$$

We can define scattering angles $\theta_n$, defined by $\alpha_n = \sin \theta_n$, which extend the definition of $\alpha_0 = \sin \theta_0$ introduced for the incident wave. Then, providing $|\alpha_n| < 1$, $\theta_n$ are real angles corresponding to propagating waves and we say that $n \in \mathcal{N}$ (note that $\mathcal{N}$ is non-empty as it always contains the zero element.) For such values of $n$, $\lambda(\alpha_n) = -i \cos \theta_n$, allowing (Q.38) to be written as

$$u^{(a)}(x, y) \sim -\frac{1}{4aG_0(\alpha_0 a; a)} \sum_{n \in \mathcal{N}} e^{i\alpha_n x} e^{i|y|\cos \theta_n}, \quad (Q.40)$$

as $|y| \to \pm\infty$. For $y > 0$, (Q.40) represents reflected plane waves propagating away from the array (located at $y = 0$) at scattering angles $\theta_n$ with amplitudes $R_n = -i/(4aG_0(\alpha_0 a; a) \cos \theta_n)$ whilst for $y < 0$, the superposition of the incident wave field implies transmitted wave amplitudes of $T_n = \delta_{n0} + R_n$. These are well-known effects in diffraction grating theory [38].

4.2.5 Defects in a doubly-periodic array of pinned points

We now move on to consider problems involving doubly-periodic arrays of pinned points. Specifically, we choose to pin an infinite elastic plate at the points $(x, y) = (na, mb)$ for $(n, m) \in \mathbb{Z}_2 \setminus \mathcal{M}$, where $a$ and $b$ denote the periodicity of the rectangular lattice in the two perpendicular directions on the plate. The set $\mathcal{M}$ represents lattice indices where defects occur, i.e. where pins are missing. In the simplest case of a single defect at the origin, $\mathcal{M} = \{(0, 0)\}$. There are two problems we can consider here. The first problem is one in which the origin is forced to oscillate with a set frequency and unit amplitude. The interest here lies in how the radiated wave energy may escape through the lattice to infinity as a function of angular frequency, $\omega$. The second is the possibility of locating trapped modes in the vicinity of the defect(s) in the lattice. These are localised wave motions which oscillate indefinitely and do not radiate energy away to infinity within the otherwise periodic lattice of pinned points.
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As we shall show, both problems require information about Bloch–Floquet waves in an infinite doubly-periodic lattice without defects, which will appear as a byproduct of our analysis.

4.2.6 Forcing of the central pin

Let us again return to a forcing problem, as the defect problem can be written as a superposition of solutions to the forcing problem over the set of defects (as shown in section 2). A general solution is written as

\[ u^{(f)}(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm}^{(f)} g(x - na, y - mb), \]  
\tag{Q.41}

where we impose the pinned conditions

\[ u_{nm}^{(f)} \equiv u^{(f)}(na, mb) = 0, \quad (n, m) \notin M, \]

\[ M = \{(0, 0)\}. \]

At the origin we set \( u_{00}^{(f)} = 1 \) to represent the forcing. Applying these conditions to (Q.41) gives

\[ \delta_{p0}\delta_{q0} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm}^{(f)} g((p - n)a, (q - m)b), \]  
\tag{Q.42}

for all \( p, q \in \mathbb{Z}. \) Multiplying this through by \( e^{-ip\theta}e^{-iq\phi} \) and summing over all \( p \) and \( q \) results in

\[ 1 = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm}^{(f)} g((p - n)a, (q - m)b) e^{-ip\theta}e^{-iq\phi}. \]  
\tag{Q.43}

Using the convolution result for Fourier series and rearranging, this can be expressed as

\[ 1 = A^{(f)}(\theta, \phi)G(\theta, \phi; a, b), \]  
\tag{Q.44}

where

\[ A^{(f)}(\theta, \phi) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm}^{(f)} e^{-in\theta} e^{-im\phi}, \]  
\tag{Q.45}

and

\[ G(\theta, \phi; a, b) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g(na, mb) e^{-in\theta} e^{-im\phi}, \]  
\tag{Q.46}

whilst the inversion formula associated with (Q.45) is

\[ a_{nm}^{(f)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} A^{(f)}(\theta, \phi)e^{in\theta}e^{im\phi} d\theta d\phi. \]  
\tag{Q.47}

The equation (Q.46) defining \( G(\theta, \phi; a, b) \) is a double lattice sum and in the present form is not suitable for computation as the series is very slowly convergent. We follow the procedure already used for a single periodic array to convert (Q.46) into a more rapidly convergent series. Thus,
we use the integral representation (Q.11) in (Q.46) to give
\[ G(\theta, \phi; a, b) = \frac{1}{8\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{G}(t, mb)e^{in(at-\theta)-im\phi} dt. \quad \text{(Q.48)} \]

Using Poisson’s summation formula for the \( n \) summation gives
\[ G(\theta, \phi; a, b) = \frac{1}{4a} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( e^{-\lambda(t_n)mb} - e^{-\gamma(t_n)mb} \right) e^{-im\phi}, \quad \text{(Q.49)} \]

where \( t_n = (\theta + 2n\pi)/a \) again. Then, reversing the order of summation in (Q.49) and summing the resulting geometric series for \( m \) gives, after some routine algebra,
\[ G(\theta, \phi; a, b) = \frac{1}{4a} \sum_{n=-\infty}^{\infty} \left( \frac{\sinh(\lambda(t_n)b)}{\lambda(t_n)} - \frac{\sinh(\gamma(t_n)b)}{\gamma(t_n)} \right) e^{-i\phi}, \quad \text{(Q.50)} \]

which is now absolutely convergent. We observe that
\[ G(\theta + 2p\pi, \phi + 2q\pi; a, b) = G(\theta, \phi; a, b), \quad \text{for } p, q \in \mathbb{Z}, \quad \text{(Q.51a)} \]
\[ G(\theta, \phi; a, b) = G(\theta, -\phi; a, b) = G(-\theta, \pm \phi; a, b), \quad \text{(Q.51b)} \]

and also that \( G \) is real-valued. Returning to (Q.44), and inverting the transform using (Q.47) gives
\[ a_{nm}^{(f)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\sinh(\lambda(t_n)b)}{\lambda(t_n)} e^{im\phi} G(\theta, \phi; a, b) d\theta d\phi. \quad \text{(Q.52)} \]

In contrast to section 4.2.2, in which coefficients were defined in terms of a single integral with a denominator \( G_0 \) which is strictly positive, in (Q.52) there is the possibility that \( G \) will vanish along curves in the two-dimensional domain of integration.

We therefore consider the implication of vanishing \( G \). Problems of this type have a long history and we direct interested readers to Martin [88] for a survey on the solution methods of Lipshitz and Koster (in the setting of the reduced wave equation), as well as section 4.9 of Lighthill [76]. The work of these authors is pertinent to the calculations we have performed here and highlight the different approaches that can be used. Applying any of these methods to our integral (51) gives the same result we have derived here.

A reworking of (Q.43) and (Q.44) in the case of a doubly-periodic array without any defects results in homogeneous versions of those equations and hence nontrivial solutions are found when
\[ G(\theta, \phi; a, b) = 0. \quad \text{(Q.53)} \]

The resulting solutions have the quasiperiodicity property in that the expansion coefficients \( a_{nm}^{(b)} \)
used in place of \( a^{(f)}_{mn} \) in (Q.41), satisfy the relation
\[
a^{(b)}_{mn} = a^{(b)}_{00} e^{i\theta} e^{im\phi} = a^{(b)}_{00} e^{i r_{nm} \cdot \mathbf{\alpha}},
\]
where \( r_{nm} \equiv (na, mb) \) are position vectors of pins in the array, \( \mathbf{\alpha} = (\alpha, \beta) \) is the Bloch wave vector and \( \theta \equiv \alpha a, \phi \equiv \beta b \). Such solutions represent Bloch–Floquet waves (hence the superscript \((b)\)). It helps to make full use of this change of coordinates to define
\[
\tilde{G}(\mathbf{\alpha}; a, b) = G(\theta, \phi; a, b),
\]
(Q.54)
defined on \(-\pi/a \leq \alpha \leq \pi/a\) and \(-\pi/b \leq \beta \leq \pi/b\), the fundamental cell of the reciprocal lattice. Solutions of (Q.53) form propagation surfaces in \((\alpha, \beta, a)\)-space which depend on the lattice aspect ratio \( b/a \). Assuming the aspect ratio \( b/a \) and dimensionless frequency \( a \) to be fixed, \( \tilde{G} \) vanishes along the curves of constant frequency on the propagation surfaces satisfying \( \tilde{G} = 0 \). If \( \tilde{G} \neq 0 \) throughout the fundamental cell of the reciprocal lattice then the frequency \( a \) is said to lie in a stop-band; wave propagation is impossible in all directions. Otherwise the frequency is said to lie in a pass band and waves can propagate throughout the infinite array.
The permissible directions of wave propagation are not defined by the direction of the Bloch wave vector \((\alpha, \beta)\) along the curves of constant frequency, but in the direction of \( \nabla \tilde{G} \equiv (\tilde{G}_\alpha, \tilde{G}_\beta) \) evaluated along those curves. This is because energy propagates in the direction of the group velocity vector, and not in the direction of the phase vector. Note that uniquely defining the phase vector outside of the unit cell is not possible, however one can observe the phase fronts of a wave inside the crystal by visual inspection [57].

To distinguish between \( \pm \nabla \tilde{G} \), which are both given by lines normal to the constant-frequency contours of the band surfaces, we can follow Lighthill [76]. From a causality argument (see also [90, 91] for a good discussion of radiation conditions), he required that energy could only radiate in a direction \( \mathbf{n} \) if \( \mathbf{c}_g \cdot \mathbf{n} > 0 \), where the group velocity \( \mathbf{c}_g \) is defined by \( d\omega = \mathbf{c}_g \cdot d\mathbf{\alpha} \), where \( \mathbf{\alpha} \) is the dimensional wave vector. From this definition, the radiation condition is equivalent to requiring \( d\omega > 0 \) when we move in the direction \( \mathbf{n} \) along the dispersion surface.

In our case, \( a = k\bar{a} \propto \omega^{1/2} \) acts as a proxy for the frequency \( \omega \) so we apply the condition that energy can propagate in a direction \( \mathbf{n} \) if \( da > 0 \) when we travel along the band surface. If we also let \( \hat{c}_g = -(\partial_a \tilde{G})^{-1} \nabla \tilde{G} = \nabla a \) be our proxy for the group velocity, this is equivalent to requiring that \( \hat{c}_g \cdot \mathbf{n} > 0 \).

Continuing, we rewrite (Q.52) as
\[
a^{(f)}_{nm} = \frac{ab}{4\pi^2} \int_{-\pi/b}^{\pi/b} \int_{-\pi/a}^{\pi/a} \frac{e^{i r_{nm} \cdot \mathbf{\alpha}}}{\tilde{G}(\mathbf{\alpha}; a, b)} d\mathbf{\alpha},
\]
(Q.55)
and note that within a stop band \( a^{(f)}_{nm} \) may be computed directly from (Q.55) and that the solution decays to zero away from the origin. Outside the stop bands, \( \tilde{G} \) vanishes along curves
of constant frequency and now the double integral in (Q.55) contains at least one line singularity within the domain of integration. Imposing the radiation condition specifies how the integrals in the inverse Fourier transform (Q.52) should be defined to interpret the effect of this line singularity.

Indeed, the far field wave behaviour is entirely determined by these line singularities and we evaluate their contribution by mapping the integral domain into an orthogonal curvilinear coordinate system which is aligned to each of the curves. The line singularities can then be processed as a continuous integral along each such line. Each integral in the variable perpendicular to the line of singularities can be deformed appropriately around the point where it crosses the singular line, which we assume is now a simple pole, according to its effect on wave radiation at infinity.

Assuming that there are \( N_c \) such curves, labelled \( C_j, j = 1, \ldots, N_c \), along which \( \tilde{G} = 0 \) in the \((\alpha, \beta)\)-plane for a given fixed frequency \( a \). We parametrise each curve by its arc length \( s \), so that \( \alpha = \alpha(s) \) with \( |\alpha'(s)| = 1 \) where the prime denotes differentiation with respect to \( s \). Using the procedure outlined above it follows that, for large \( |\mathbf{r}_{nm}| \),

\[
a^{(f)}_{nm} \sim \frac{ia \beta}{2\pi} \sum_{j=1}^{N_c} \int_{C_j} e^{i\mathbf{r}_{nm} \cdot \alpha(s)} \mathbf{n}(s) \cdot \nabla \tilde{G} \, ds,
\]

where \( \mathbf{n}(s) = (\beta'(s), -\alpha'(s)) \) denotes the unit normal to the curve \( C_j \). For later convenience, we also define a vector \( \hat{\mathbf{n}}(s) = \mu(s) \mathbf{n}(s) \), where \( \mu(s) = \pm 1 \), so that \( \hat{\mathbf{n}} \) always points outwards from the origin. In the above, we evaluate the integrals with respect to the coordinate parallel to \( \mathbf{n}(s) \) first. If \( \hat{\mathbf{n}} \cdot \hat{\mathbf{c}}_g > 0 \), the pole represents energy propagating away from the origin, so we deform the contour and complete it in such a way that it encloses the singularity. However, if \( \hat{\mathbf{n}} \cdot \hat{\mathbf{c}}_g < 0 \), then energy is travelling towards the forced pin, and we deform and complete the integration contour without enclosing the pole. (In this work we have assumed that we are not at a saddle point of the dispersion surface, where \( \hat{\mathbf{n}} \cdot \hat{\mathbf{c}}_g = 0 \); in such a case the residue would be more complicated than those in equation (Q.56). Moreover, the application of the stationary phase approximation below would need to be adjusted, and would require the use of Airy functions: see the treatment of caustics in section 4.11 of Lighthill [76].)

The next step in determining the dominant contribution as \( |\mathbf{r}_{nm}| \to \infty \) is to identify stationary phase points in the oscillatory integral. These occur at \( P_j \) points on the curve \( C_j \) given by \( s = s_{jk} \) when

\[
\mathbf{r}_{nm} \cdot \alpha'(s_{jk}) = 0, \quad k = 1, \ldots, P_j, \quad (Q.57)
\]

That is, stationary points are those points on the curve \( \tilde{G} = 0 \) where the radial vector to the point \( \mathbf{r}_{nm} \) is perpendicular to that curve, implying \( \mathbf{r}_{nm} = |\mathbf{r}_{nm}| \hat{\mathbf{n}}(s) \). Applying the standard result for the method of stationary phase gives

\[
a^{(f)}_{nm} \sim \frac{ia \beta}{\sqrt{2\pi}} \sum_{j=1}^{N_c} \sum_{k=1}^{P_j} e^{i\mathbf{r}_{nm} \cdot \alpha(s_k) e^{i\text{sgn}(\gamma(s_k))\pi/4}} |\gamma(s_k)|^{1/2} \mathbf{n}(s_k) \cdot \nabla \tilde{G} \bigg|_{s=s_{jk}} \quad (Q.58)
\]
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where \( \gamma(s) = r_{nm} \cdot \alpha''(s) = |r_{nm}|\kappa(s) \), and \( \kappa(s) = \mu(s)(\beta'(s)\alpha''(s) - \alpha'(s)\beta''(s)) \) is the curvature.

Hence we may write (Q.58) as

\[
a_{nm}^{(f)} \sim \frac{ia b}{\sqrt{2\pi |r_{nm}|}} \sum_{j=1}^{N_n} \sum_{k=1}^{P_n} e^{i r_{nm} \cdot \alpha(s) e^{i \text{sgn}(\kappa(s)) \pi/4}} |\kappa(s)|^{1/2} n(s) \cdot \nabla \tilde{G} \big|_{s=s_{jk}}.
\]

Note that if there are no stationary points then the summation above equates to zero since we are not attempting to characterise any behaviour in the far-field which decays more rapidly than \( |r_{nm}|^{-1/2} \). Consequently, we have not reconstructed the wave field at infinity, but merely provided an argument for directions of wave radiation and the asymptotic form for the coefficients \( a_{nm}^{(f)} \) at infinity in directions where wave radiation is permitted. Specifically, radiation is possible in those directions that cross a dispersion curve parallel to the group velocity vector at the crossing point, and where the group velocity vector also points away from the forced pin.

4.2.7 Homogeneous defect problem

We now consider the possibility of finding trapped modes, or localised wave motions, due to multiple defects in a doubly-periodic lattice of pinned points. A general solution can be written as

\[
u^{(d)}(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm}^{(d)} g(x - na, y - mb),
\]

and we specify \( a_{nm}^{(d)} = 0 \) for \( (n, m) \in \mathcal{M} \) which represents that there is no contribution to the displacement from the set \( \mathcal{M} \) of pins that have been removed from the lattice. This is a homogeneous problem with no forcing, and we seek nontrivial coefficients \( a_{nm}^{(d)} \) for \( (n, m) \notin \mathcal{M} \) satisfying the pin conditions \( u^{(d)}(pa, qb) = 0 \) for \( (p, q) \notin \mathcal{M} \). Letting \( u_{pq}^{(d)} = u^{(d)}(pa, qb) \) represent the unknown plate displacements for \( (p, q) \in \mathcal{M} \) we can write

\[
u^{(d)}(x, y) = \sum_{(p,q)\in\mathcal{M}} u_{pq}^{(d)} u^{(f)}(x - pa, y - qb).
\]

Thus we can see that \( u^{(d)}(pa, qb) = 0 \) if \( (p, q) \notin \mathcal{M} \), and that

\[
a_{nm}^{(d)} = \sum_{(p,q)\in\mathcal{M}} u_{pq}^{(d)} a_{n-p,m-q}^{(f)},
\]

so if \( (n, m) \in \mathcal{M} \)

\[
0 = \sum_{(p,q)\in\mathcal{M}} u_{pq}^{(d)} a_{n-p,m-q}^{(f)}.
\]

Thus, we have a homogeneous system for the unknown displacements \( u_{pq}^{(d)} \) at the defects. In other words, the determinant of the matrix

\[
K_{nmpq} = a_{n-p,m-q}^{(f)}, \quad (n, m), (p, q) \in \mathcal{M},
\]

(Q.64)
is required to vanish for trapping solutions to exist (since we require that $u_{pq}^{(d)}$ for $(p,q) \in \mathcal{M}$ are not all zero). The Toeplitz matrix $K$ is sized $N \times N$, where $N$ represents the number of defect points contained in the set $\mathcal{M}$. Computing the kernel of $K$ allows us to calculate the corresponding trapped mode(s) from (Q.61). If the kernel of $K$ were empty, the $u_{pq}^{(d)}$ would all be zero, and the lattice may as well have been pinned periodically with no defects. Then we would return to the Bloch–Floquet problem discussed earlier where trapping solutions cannot exist.

To compute the shape of the trapped modes inside the defect we need to compute the non-zero $u_{pq}^{(d)}$ given in (Q.63). This is done by first determining the kernel of the Toeplitz matrix $K$ from which we can then compute the scattering coefficients in (Q.62) allowing us to construct the displacement (Q.60) after suitable truncation of both sums. In practice this can be done by finding the eigenvector(s) of $K$ with eigenvalue zero.

Since $G$ is real, if we work at frequencies $\alpha$ that lie in a stop band (so that $G$ does not vanish in the domain integral of (Q.52)) then we are assured that $\lambda(t_0)$ is real and hence the determinant of $K_{nm}$ is also real. Thus, the task of finding trapping solutions is simply one of finding real frequencies that force a real determinant to vanish. We note that the realness and symmetry of $K_{nm}$ implies that the number of linearly independent trapped modes is equal to the multiplicity of the zero eigenvalue.

It is instructive to consider the case where there is only a single defect at the origin, so that $\mathcal{M} = \{(0,0)\}$ and then from (Q.62) the requirement for a trapped mode (Q.52) is simply

$$0 = a_{00}^{(f)} = \frac{1}{4\pi^2} \int_0^{\pi} \int_0^{\pi} \frac{1}{G(\theta, \phi; a, b)} d\theta d\phi. \quad (Q.65)$$

Clearly, for solutions of (Q.65) to exist, we require $G^{-1}$ to take different signs in the domain $0 < \theta, \phi < \pi$. Since $\lambda(t_0) = -i(1-\theta^2)^{1/2}$ where $t_0 \in (0,1)$ we see immediately that $G^{-1}(\theta, \phi; a, b) = 0$ along the circle $\theta^2 + \phi^2 = 1$. This provides a good motivation for seeking a trapped mode solution. For larger values of $\alpha$, where $|t_n| < 1$ for values of $n$ other than zero, $G^{-1}$ also vanishes along curves $\phi^2 + (\theta + 2n\pi/a)^2 = 1$, but such arguments are redundant as numerical results suggest that such values of $\alpha$ lie outside a stop band.

### 4.2.8 Lines of defects: Fabry–Perot resonances

Assume now that entire rows are left unpinned in an otherwise doubly-periodic array of pinned points. For such a configuration we seek solutions which are trapped by the lines of defects and rapidly decay away from the defect lines. Such solutions are often referred to as Fabry–Perot resonances, or waveguide modes. The missing pins are assumed to lie along the rows $y = mb$ where $m \in \mathcal{M} \subset \mathbb{Z}$. In the simplest case where a single line of pins are removed along $y = 0$,
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\( \mathcal{M} = \{0\} \). The general solution can be written as in (Q.60) with \( a_{nm}^{(d)} = 0 \) for \( n \in \mathbb{Z} \) and \( m \in \mathcal{M} \).

Clearly there is no periodicity in the \( y \)-direction of the array, however it remains periodic in the \( x \)-direction. Consequently any solution must be quasiperiodic in \( x \); that is, they must satisfy

\[
\psi^{(d)}(x,y) = \sum_{m=-\infty}^{\infty} a_{0m}^{(d)} e^{i\theta} g(x - na, y - mb),
\]

where, unlike in previous sections, \( \theta \) is a freely-chosen parameter which reflects the quasiperiodicity of the solution whilst \( p \in \mathbb{Z} \). Consequently \( a_{nm}^{(d)} = a_{0m}^{(d)} e^{i\theta} \) (which can be shown by examining (Q.66) in terms of (Q.60)) and therefore (Q.60) becomes

\[
\psi^{(d)}(x,y) = \sum_{m=-\infty}^{\infty} a_{0m}^{(d)} e^{i\theta} g(x - na, y - mb).
\]

We then apply the pinned conditions \( \psi_{0q}^{(d)} = \psi^{(d)}(0,qb) = 0 \) for \( q \notin \mathcal{M} \), which only has to be made along \( x = 0 \), since all other values of \( x \) have been accounted for by (Q.66). This gives

\[
\sum_{m=-\infty}^{\infty} a_{0m}^{(d)} \sum_{n=-\infty}^{\infty} e^{-in\theta} g(na, (q - m)b) = \sum_{r \in \mathcal{M}} u_{0r}^{(d)} \delta_{qr} \quad \forall q \in \mathbb{Z},
\]

where the \( u_{0q}^{(d)} \) \( (q \in \mathcal{M}) \) are the unknown plate displacements along \( (x,y) = (0, qb) \) (i.e. along the cross-section \( x = 0 \) perpendicular to the line of defects). Multiplying (Q.68) by \( e^{-iq\phi} \) and summing over all \( q \) transforms (Q.68) into

\[
A^{(d)}(\phi) G(\theta, \phi; a, b) = \sum_{q \in \mathcal{M}} u_{0q}^{(d)} e^{-iq\phi},
\]

where we now have

\[
A^{(d)}(\phi) = \sum_{m=-\infty}^{\infty} a_{0m}^{(d)} e^{-im\phi},
\]

and \( G \) is the double-lattice sum for the infinite periodic array defined as in (Q.46). The \( \theta \)-dependence is implicit in \( A^{(d)}(\phi) \) and \( u_{0q}^{(d)} \).

Rearranging (Q.69) for \( A^{(d)}(\phi) \), inverting the transform defined in (Q.70) to return to \( a_{0m}^{(d)} \), and then applying the conditions \( a_{0m}^{(d)} = 0 \) for \( m \in \mathcal{M} \) gives the homogeneous system of equations

\[
0 = \sum_{q \in \mathcal{M}} u_{0q}^{(d)} \int_{-\pi}^{\pi} e^{i(m-q)\phi} G(\theta, \phi; a, b) d\phi, \quad m \in \mathcal{M}.
\]

Thus, for a Fabry–Perot resonance or trapped mode, we require the determinant of the Toeplitz matrix

\[
K^{(d)}_{mq} = \int_{0}^{\pi} \frac{\cos((m - q)\phi)}{G(\theta, \phi; a, b)} d\phi, \quad m, q \in \mathcal{M},
\]

to vanish. We note again that \( G \) is real, so provided it is also non-zero for \( 0 < \phi < \pi \) for a given value of \( \theta \), the integral (Q.72) is real and the determinant of the matrix defined by (Q.72) is also real.
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4.2.9 Results and discussion

4.2.10 Forcing and scattering in one-dimensional arrays

We begin by considering the one-dimensional array problem of a forced pin (a defect subject to time-harmonic forcing of unit amplitude), which is shown for an array of period $a = 2\pi$ in Figure 4.1(a). In this figure we can see clear left-right and up-down symmetry in the computed field, as well as clear energy radiation as we move away from the forcing location at the origin. As outlined in section 4.2.4, we show that it is possible to express the unknown coefficients for the scattering problem in terms of the coefficients for the forcing problem. This leads us to Figure 4.1(b), where the total field is computed for a particular scattering problem. A plane wave is normally-incident on the array from $y > 0$, so that $\theta_0 = 0$ with $a = \pi$. In this configuration no other diffraction orders are excited, and so we can determine the proportion of energy reflected and transmitted from the array straightforwardly. Here they are given by $R = |R_0|^2 = 0.67801$ and $T = |T_0|^2 = 0.32199$ (defined in section 4.2.4).

Thus the symmetry of the plate displacement about the line of pins shown in Figure 4.1(b) is due to it being a particular snapshot in time of the superposition of a totally transmitted plane wave and a symmetric circular outgoing wave from the defect. Vertical channels of minimal displacement can be observed both above and below this single defect. We also observe a focusing of energy at the origin where the pin has been removed.

For a plane wave with incident angle $\theta_0 = \pi/6$, Figure 4.2(a) shows a snapshot in time of a different scattered wave pattern. There is again a defect at the origin. In this plot we can clearly see the formation of partial standing waves above the grating, caused by a large amount of reflection of wave energy by the array (as previously, no other diffraction orders are excited and so $R = |R_0|^2 = 0.87550$ and $T = |T_0|^2 = 0.12449$). However below the grating we can see the interaction effect between the transmitted wave energy and circular waves emanating from the defect, with occasional destructive interference being produced to the left of the defect.

Using our formulation we can also consider the effects of multiple pins being removed, as shown in Figure 4.2(b) where for a single array of period $a = 2$, four defects have been removed to recreate a double-slit experiment. For this problem we consider an incident wave at $\theta_0 = \pi/12$, which corresponds to a higher level of energy transmission ($R = |R_0|^2 = 0.57652$ and $T = |T_0|^2 = 0.42348$). Here the circular waves emanating from the defects also have a strong effect, with a clear channel of minimal displacement directed downwards and to the right from the right hand defect.
4.2.11 Forcing and band surfaces for doubly periodic arrays

We now consider the problem of determining the band surfaces of our pinned plate, which allows us to determine information about the propagating modes, or Bloch modes, that are supported by our doubly periodic medium [57]. From this, the band surfaces also reveal the locations of full or partial stop bands, which correspond to frequencies at which no propagation through the array is possible, for given \((\theta, \phi)\) values. It is in these stop bands that we look for trapping behaviour when defects are introduced.

Movchan et al. [106] considered the particular case of a square array of pins \((b/a = 1)\) by taking limits of a more general system based on multipole expansions designed to investigate
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Figure 4.2: (a): Total field ($\text{Re}\{u^{(s)}\}$) demonstrating scattering by a plane wave at angle $\theta_i = \pi/6$ on a defective grating with $a = \pi$ and $M = \{0\}$. (b): Total field ($\text{Re}\{u^{(s)}\}$) demonstrating scattering by a plane wave at angle $\theta_i = \pi/12$ incident on a single array of period $a = 2$ with a four defects at index locations $M = \{-4, -3, 3, 4\}$.

the band-gap structure of periodic arrays of finite-radius circular holes. Thus, when the radius of a clamped hole tended to zero, a simplified dispersion relation was derived in terms of lattice sums for the Helmholtz and modified Helmholtz equations (see equation 6.2 of [106]). As outlined in their paper and in [78], the computation of these lattice sums is extremely complicated. Here, we have derived an alternative dispersion relation (Q.53) in which the lattice sum (Q.50) is both convergent and simple to compute. Our solutions coincide with those in Movchan et al. [106], who were also able to determine that the band surfaces for arrays of pins were bound between singularities arising from these lattice sums. These singularity curves correspond to discrete values at which plane wave solutions (with the quasiperiodicity property (Q.66)) would be supported in the medium in the absence of our pins [50]. This acts as a useful tool for computing solutions of (Q.53) as an upper and lower bound for each zero can be determined.
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straightforwardly.

To determine the band surfaces from (Q.53) we search for values of \( a \) (which is a dimensionless proxy for the wavenumber) for given values of \( \theta \) and \( \phi \) (which are themselves proxies for Bloch vector elements). For the case of a square array \((b/a = 1)\) we show the first, second and third band surfaces in Figures 4.3(a), 4.3(b) and 4.3(c) (respectively) over a quarter of the Brillouin zone. A full picture of the band surface over the entire Brillouin zone can be obtained by reflection in the \( \theta \) and \( \phi \) axes. From the picture of the first band surface we can deduce that we have a complete stop band from \( 0 < a < \pi \), and by comparing Figures 4.3(a) and 4.3(b), that only partial stop bands exist between the first and second band surfaces (i.e. the stop band frequency depends on \( \theta \) and \( \phi \)). Movchan et al. [106] show that only partial stop bands exist between all other higher order band surfaces. Additionally, we note that the width of the first stop band is proportional to the spacing ratio \( b/a \), that is, the first stop band exists in the range \( 0 < a < \pi/(b/a) \) for rectangular arrays, as outlined in section 4.2.6.

In section 4.2.6 it was shown that wave radiation through the array from a forced pin can only occur when the Bloch vector \( \alpha \) and the vector tangent to a constant frequency curve on the dispersion surface are at right angles, and provided the gradient \( \nabla a = \tilde{c}_q \) of the dispersion surface also points away from the origin at any such points. Thus the amplitude of the wave radiated to infinity is inversely proportional to both the magnitude of that gradient and to the square root of the curvature along a curve of constant frequency.

Now, since \( b/a = 1 \), directions in \( \alpha \)-space are the same as in \((\theta, \phi)\)-space. Moreover, if wave propagation is possible in a certain direction of \( \alpha \)-space, then it is also possible in the same direction of \((x, y)\)-space. Thus, observing Figures 4.3(a) and 4.3(b) we can say the following. For \( 0 < a < \pi \) there is no wave radiation to infinity as the frequency lies in a stop band. Then, as \( a \) increases from \( \pi \) to approximately 3.6238 (the level of the saddle point) there is still no wave radiation as the only possible directions for wave propagation are along \( x = 0 \) (in both directions) and \( y = 0 \) (in both directions), but there the gradient of the dispersion surface along outgoing lines is negative. As \( a \) increases beyond 3.6238, again the lines \( x = 0 \) and \( y = 0 \) are excluded from wave radiation as the gradient is negative but wave radiation along \( x = y \) (and by reflection along \( x = -y \)) is allowed, determined by values of \( \theta \) and \( \phi \) in the portion of the dispersion surface for which the gradient is positive. The amplitude factor of radiated waves is large for \( a \) close to the value at the saddle point where the gradient is close to zero (here the group velocity is small and so energy only propagates slowly away from the origin) and decreases as \( a \) increases up to the the point at which we switch from the first band to the second band \((a \simeq 4.4429)\). According to the second band, all possible directions of wave radiation are associated with negative gradients on the dispersion surface and therefore there is no wave radiation to infinity associated with this band. However, moving on to Figure 4.3(c), we can see that the third band also starts at \( a \simeq 4.4429 \) and in this case there are possible directions of wave radiation to infinity along \( x = 0 \) and \( y = 0 \) for values between \( a \simeq 6.3137 \) and \( a \simeq 7.0248 \) where the gradients of the dispersion surfaces are increasing.
The overall picture is one in which the scope to radiate waves from an oscillating source at the origin to infinity is limited, even if the frequency is inside a band surface. However, we have shown that oscillations within certain higher frequency ranges can send waves through the lattice in different directions (here either along $x = y$ for $3.6238 < a < 4.4429$ or along the axes if $6.3137 < a < 7.0248$).

In Figure 4.4 we show a snapshot in time of the plate displacement when a central pin in a doubly-periodic pinned plate is made to oscillate with unit amplitude at a dimensionless value of $a = 1$ and spacing ratio of $b/a = 1/2$. This corresponds to a frequency within the stop band for the doubly-periodic array ($0 < a < 2\pi$ when $b/a = 1/2$) and hence the figure confirms that there is no energy propagation to infinity. Indeed, the displacement is predominantly contained horizontally within one layer of pins, and contained vertically within two layers.

### 4.2.12 Trapped modes in doubly periodic arrays with defects

We now examine the different mode shapes that can be supported inside defects which exist in doubly periodic square arrays. We look for the resonant frequencies of these defects inside the first stop band, which for square arrays ($b/a = 1$) is the interval $0 < a < \pi$. Each trapped mode corresponding to a non-degenerate frequency was also confirmed by exciting a large finite cluster, with the same defect in its centre, with an incident wave. At degenerate frequencies, different combinations of the two possible trapped modes appeared, depending on the angle of incidence and the relative closeness of the frequencies. It would be interesting to attempt to predict how the excited modes depend on this angle, but this is not attempted here.

We begin by determining the resonant frequencies for two problems – firstly, a single defect at the origin, and secondly, a $3 \times 3$ sized defect cluster centred about the origin where pins corresponding to the indices $(-1,0,1) \times (-1,0,1)$ are removed. This is done by evaluating the determinant of the Toeplitz matrix $K_{mnpq}$ as given in (Q.64) for varying $a$. We can see from Figure 4.5(a) that only one resonant frequency exists in the first stop band for the single defect problem and is given by $a \approx 2.53727$. This compares well with the estimate $a \approx 2.538$ given in McPhedran et al. [99] which was evaluated by computing the kernel of a truncated matrix of standard Green’s functions (of the form given in (Q.7)).

For the $3 \times 3$ defect we can see from Figure 4.5(b) that there are resonant frequencies at $a \approx 1.35692, 1.95271, 2.38206, 2.61818, 2.62461$ and $a \approx 2.93110$. In this case we have the added complication of degenerate (repeated) roots corresponding to the determinant curve touching the zero axis as opposed to crossing it completely. These degenerate frequencies are a consequence of the symmetry of the geometry and the associated trapped mode. That is, it is possible for two modes to exist for the same frequency $a$, one with a line of symmetry in $x = 0$ and the other in $y = 0$. If $a/b \neq 1$ then modes symmetric in $x$ are different to modes symmetric in $y$, eliminating the presence of repeated roots.
Figure 4.3: (a): Contour plot of \( a \) values constituting the first band surface, over 1/4 of the Brillouin zone, for a doubly periodic square array of pinned points \( b/a = 1 \). Figures (b) and (c) show the second and third band surfaces (respectively) over 1/4 of the Brillouin zone.

The discontinuity of the determinant curve in Figure 4.5(b) at \( a = 2.82743 \) is associated with the determinant of \( K \) becoming infinitely large. This is a consequence of the Green’s function being undefined at this value of \( a \) for all \( 0 < \theta < \pi \) (when \( b/a = 1 \)), and does not correspond to
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a trapped mode.

Returning to the single defect problem, the mode shape corresponding to \( a \approx 2.53728 \) is computed in Figure 4.6 which shows one peak, symmetric about both \( x = 0 \) and \( y = 0 \) which is well trapped inside the defect with negligible displacement throughout the surrounding array.

For the \( 3 \times 3 \) defect the associated modes are given in Figure 4.7. A number of these mode shapes have clear reflection and rotational symmetry inside the defect, most notably those corresponding to the degenerate frequencies which are given in Figures 4.7(b), 4.7(e), and 4.7(f). Some of the mode shapes computed here are reminiscent of the mode shapes for square plates that are completely clamped, as discussed in [73]. Again, we observe a rapid decay of displacement with distance away from the defect into the array in all figures.

We finally consider a \( 4 \times 1 \) defect in a square array \( (b/a = 1) \), for which all trapped mode frequencies are distinct (i.e. not repeated roots) in the first stop band. The corresponding mode shapes are given in Figure 4.8 and bear resemblance to the first few harmonics for waves on a string, which is to be expected given the slender geometry of our defect cluster.

4.2.13 Fabry–Perot line defects in two-dimensional arrays

For the waveguide problem, we remove entire rows of pins from the otherwise perfect doubly periodic array. We then vary our quasiperiodicity parameter \( \theta \) and look for values of \( a \) that satisfy the relation given by (Q.72), which essentially reveals the frequencies at which different waveguide modes are supported inside the defect(s). These waveguide modes transport wave energy inside the line defects without leakage into the surrounding array. There may still be some motion outside the defect but its amplitude rapidly decays with distance away from it.
In Figure 4.9(a) we examine (Q.72) for the case of a single line defect \((m = 0)\) for multiple aspect ratios \(b/a\). We can see that for this single line defect \((b/a = 1)\) we have a single curve which exists in the interval \(1.96720 < a < \pi\). We see similarly sloped curves for \(b/a = 1.5\) and \(b/a = 2\), which end abruptly when they approach the edge of their first stop bands (for example, when \(b/a = 2\), the stop band interval is \(0 < a < \pi/2\)).

In Figure 4.9(b) we consider the case when we have two non-neighbouring line defects \((m = 0, -3)\) for multiple aspect ratios \(b/a\). For \(b/a = 1\), (Q.72) reveals two unique values of \(a\) over an interval of \(\theta\) values, which corresponds to one symmetric mode and one antisymmetric mode supported across both line defects (represented by the broken and solid lines respectively). In a small interval near \(\theta \simeq 2.25\) the waveguide mode frequencies \(a\) become nearly degenerate but nevertheless two distinct modes always exist. The curves for \(b/a = 1.5\) and \(b/a = 2\) are similar,
but the near-degeneracy occurs at lower values of $a$ and $\theta$, and there is no clear splitting near the approach to the first stop band when compared to the $b/a = 1$ curve.

For the single line defect problem, the waveguide mode corresponding to $b/a = 1$, $a = 2.1$, $\theta = 0.87850$ is computed using (Q.67) and shown in Figure 4.10(a). This mode is well contained with only small troughs existing outside the defect when the mode itself is at a minimum. For the doubly periodic problem, the waveguide mode corresponding to $b/a = 1$, $a = 2.1$, $\theta = 0.75567$ is given in Figure 4.10(b) and in Figure 4.10(c) the mode corresponding to $\theta = 0.97115$ is shown. From Figure 4.10(b) we can take a vertical slice through the field and determine that this first mode is antisymmetric, and as before, we have some displacement occurring outside the array corresponding to minima in the mode, with little interaction in the space between the two channels. For Figure 4.10(c) we have the second waveguide mode which is symmetric and demonstrates minimal displacement outside the line defects for $x < 0$. However, there is now a strong interaction between the two channel defects.

4.2.14 Conclusions

In this paper we have examined a number of problems connected with defects in one- and two-dimensional rectangular arrays of periodically pinned plates. A method has been outlined for analytically determining a variety of properties of the solution including the plate displacement for problems where a single pin within the array is forced to oscillate and where incident waves are diffracted by a one-dimensional pinned array in which multiple pins are removed. For two-dimensional arrays, we have also provided analytical expressions to determine certain localised modes which exist when either a finite number of pins are removed or entire rows are removed. Connections between forced pin problems and scattering and trapping problems in which pins are removed have been highlighted in both one- and two-dimensional situations. This has similarities
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Figure 4.7: Mode shapes that are supported inside $3 \times 3$ defect in doubly periodic domain ($b/a = 1$) at (a): $a \simeq 1.35692$, (b): $a \simeq 1.95271$, (c): $a \simeq 2.38206$, (d): $a \simeq 2.61818$, (e): $a \simeq 2.62461$ and (f): $a \simeq 2.93110$.

We have also shown theoretically how to determine the discrete directions at which waves may radiate through a doubly-periodic array in which the central pin is forced to oscillate and shown that different frequency ranges lead to wave radiation in different directions. In addition, we derived the asymptotic amplitudes of the source strengths in these directions.

There are a number of interesting extensions to the current work that could be made. For example, the problem of plane wave scattering by either one- or two-dimensional semi-infinite periodic arrays of pins could be solved using the discrete Wiener-Hopf technique as done by [79] and [154] for scattering of long waves by cylinders. Defects (particularly line defects) could also...
be put introduced into their formulations in a relatively straightforward manner.

Another straightforward extension of the methods presented here can be made for problems involving defects in a semi-infinite elastic plate which is pinned periodically along its free edge. The solution for the corresponding problem without defects has been presented in [37] which simply requires that the Green’s function representing the effect of each pin be altered, to take account of the free-edge conditions [49].
Figure 4.9: (a): Plot showing the values of $a$ for which $\det \{K_{m,q}\} = 0$ for a single line defect ($m = 0$). (b): Values of $a$ when $\det \{K_{m,q}\} = 0$ for the two line defect case ($m = 0, -3$) where $b/a = 1$ (blue curve), $b/a = 1.5$ (black curve), $b/a = 2$ (red curve).
Figure 4.10: (a): Waveguide mode $\text{Re}\{u^{(d)}\}$ for a single line defect at $\theta = 0.87850$ for $a = 2.1$, $b/a = 1$.
(b-c): Waveguide modes for a double line defect corresponding to (b) $\theta = 0.75567$ and (c) $\theta = 0.97115$ with $a = 2.1$ and $b/a = 1$. 
4.3 Discussion

In this chapter we have examined several defective pinned PlaC structures, from the removal of pins in an otherwise perfect one-dimensional array to the design of PlaCs which feature multiple waveguides.

Using the framework in Chapter 3, one could construct a large finite cluster of pins which feature these defects. It would then be possible to see the different modes that are excited by sending in a normally incident Gaussian beam at the prescribed frequencies. This has been done as a check on numerics and gives identical mode shapes to the ones calculated here, but this is not included in this thesis for compactness.

We now proceed to the case of arbitrarily shaped scatterers embedded in thin elastic plates.
5

Arbitrarily shaped scatterers in elastic plates

5.1 Introduction

In this chapter we consider the problem of wave scattering by cavities of arbitrary geometry. Previously we restricted our attention to wave scattering by circular inclusions and pins, however we are interested in investigating more complex geometries, such as elliptical scatterers. As discussed earlier, it is possible to obtain closed form solutions for particular geometries, but the numerical computation of basis functions (such as Mathieu functions for the ellipse), can be a difficult task. We aim to overcome this and provide a robust procedure for determining the wave scattering behaviour of an arbitrarily shaped scatterer, as well as one- and two-dimensional arrays of arbitrarily shaped scatterers. An outline of the three different problems we consider in this chapter can be seen in Figure 5.1.

This chapter is comprised of two publications. The first paper “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems” examines the problem of a single arbitrary cavity in a thin elastic plate, and the paper “Flexural wave filtering and platoonic polarisers in thin elastic plates” considers one- and two-dimensional arrays constructed of scatterers which are of arbitrary geometry.
5.2 Single arbitrarily shaped scatterer

The first paper (which addresses the problem of a single inclusion) was motivated by the work of Bennetts and Williams [12] who considered the behaviour of floating sea ice containing a single arbitrarily shaped hole. They considered the case when a plate was suspended above an incompressible fluid medium and considered a finite element solution for the displacement (with free-surface and ice-covered modes as the basis). We examined a simplification of this problem by assuming that the plate was surrounded above and below by a vacuum, and were able to express this in-vacuo problem as a system of boundary integral equations. These were then solved using boundary element methods. In a continuation of this work the second paper considered both one- and two-dimensional arrays of clamped scatterers, which are of arbitrary geometry, utilising the solution procedures outlined in Botten et al. [16], Movchan et al. [107].

5.2 Single arbitrarily shaped scatterer

The paper “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems” examines the problem of a single arbitrary scatterer which is subject to clamped, free-edge and simply-supported boundary conditions at the edge. We also examine the related vibration problem for arbitrarily shaped plates. It is important to note that from a practical perspective, the simply-supported cavity problem considered here may be non-physical, or cor-

Figure 5.1: The three different scattering problems considered here: (a) a single arbitrarily shaped scatterer, (b) a single array of arbitrarily shaped scatterers, and (c) a doubly periodic square array in an infinite plate.
respond to a complicated configuration which is difficult to implement [151]. That said, it is possible to impose simply-supported edge conditions mathematically (as they correspond to a vanishing displacement and bending moment at the edge).

There are certain differences in the text compared to the main body of the thesis; the largest of these is the use of the alternative term ‘anti-Helmholtz’ to refer to the modified Helmholtz equation. Consequently any terms with the superscript A are equivalent to the superscript M (i.e., \(G^A \equiv G^M\)).

Additionally, to reduce complexity we do not decompose the displacement into \(w^H\) and \(w^M\) components. It is possible to specify \(w = w^H + w^M\) into equations (R.9a) and (R.9b) of the present paper to obtain the system

\[
\begin{align*}
\frac{1}{2} \frac{\partial w^M}{\partial s'} &= \int_{\partial \Omega} \left\{ \partial_{n'} G^M w^M - \partial_{n'} w^M G^M \right\} dS', \\
\frac{1}{2} \frac{\partial w^H}{\partial s'} &= w^H_i + \int_{\partial \Omega} \left\{ \partial_{n'} G^H w^H - \partial_{n'} w^H G^H \right\} dS',
\end{align*}
\]

for the case of a single inclusion, after making use of the identities \((\Delta + k^2) w^M = 2k^2 w^M\) and \((\Delta - k^2) w^H = -2k^2 w^H\).

However after decomposing the displacement, one must also rewrite the boundary conditions in a similar form. For clamped-edge conditions this is straightforward, but for more complicated boundary conditions such as free-edge and simply-supported edge conditions, the process is highly convoluted and can lead to highly intractable expressions.

There are also the following deviations from standard notation:

i. We consider one type of incident wave type \(w^I = w^H_i\), where we specify unit amplitude \((\delta_0 = \sqrt{|\chi_0|})\) with \(\theta_i = \pi/2\).

ii. We define \(\gamma\) as the variable used to parameterise the boundary with respect to arc length.

Note that for the case of a single inclusion, we do not consider modified Helmholtz incident waves \(w^M_i\).

### 5.3 Publication

“Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems”
Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems

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Abstract: This paper presents a solution for the displacement of a uniform elastic thin plate with an arbitrary cavity, modelled using the biharmonic plate equation. The problem is formulated as a system of boundary integral equations by factorizing the biharmonic equation, with the unknown boundary values expanded in terms of a Fourier series. At the edge of the cavity we consider free-edge, simply-supported and clamped boundary conditions. Methods to suppress ill-conditioning which occurs at certain frequencies are discussed, and the combined boundary integral equation method is implemented to control this problem. A connection is made between the problem of an infinite plate with an arbitrary cavity and the vibration problem of an arbitrarily shaped finite plate, using the jump discontinuity present in single-layer distributions at the boundary. The first few frequencies and modes of displacement are computed for circular and elliptic cavities, which provide a check on our numerics, and results for the displacement of an infinite plate are given for 4 specific cavity geometries and various boundary conditions.

5.3.1 Introduction

The problem of computing the vibration of thin elastic plates of finite size subject to various boundary conditions has been widely studied (Graff [46], Itao and Crandall [56], Leissa [73], Leissa and Narita [74]). These papers have been focused primarily on determining the modes of vibrations. By comparison the problem of wave scattering by infinite plates has received considerably less attention. This is in contrast to the case for membranes, where the problem of scattering for the Helmholtz equation is nearly as well-studied as the eigenvalue problem for the Laplacian. The simplest problem which can be considered for the infinite plate is wave scattering by a single circular cavity. This was considered by Konenkov [67], Norris and Vemula [111] and recently has been extended to an array of circular cavities by Movchan et al. [106] and Parnell and Martin [115]. The problem of wave scattering by a cavity is strongly connected with crack problems in plate and shell theory, since a crack may be thought of as a degenerate cavity. However it appears that only straight line cracks, and not cracks of arbitrary shape have been considered (Andronov and Belinskii [5], Norris and Wang [112], Porter and Evans [120]).

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If the geometry of the cavity is circular then the solution can be found by separation of variables, in terms of Bessel functions. Similarly, if the cavity is elliptical a solution can be found in terms of Mathieu functions. However, for arbitrary boundaries this method is no longer applicable, and an alternative numerical solution method must be developed. There exists a need to investigate more varied geometries for the case of a single cavity, which can open the way to studies of clusters and arrays of diverse cavities. In the context of solving the problem of water-wave scattering by polynya (holes in floating sheets of ice), Bennetts and Williams [12] solved the problem of water-wave scattering by an arbitrarily shaped cavity in a plate floating on water. They factorised the problem into a system of integral equations and expanded the boundary values in a Fourier series. The present work is closely based on their work, although they did not consider the phenomena of irregular frequencies.

It is well known that the solution for Helmholtz’s equation by boundary integral equations leads to certain irregular frequencies, for which the matrices are ill-conditioned (Kleinman and Roach [64, 65]). This phenomenon occurs in other boundary integral approaches, for example, in fluid/structure interactions (Linton and McIver [80]). This ill-conditioning is a feature of the solution method, and it is associated with an interior eigenvalue problem. Various methods have been developed to control these irregular frequencies, and a popular method is the combined boundary integral equation method (CBIEM). This method involves overdetermining the standard system of integral equations with a modified system of integral equations evaluated at a series of field points taken to be inside the cavity region. This method was used by Lau and Hearn [72] to compute the added mass and damping coefficients for a three-dimensional wetted ship hull. Other methods for controlling ill-conditioning exist, including the null-field method outlined by Martin [87], and methods based on modifying the integral operator and its domain (Lau and Hearn [72]).

This paper presents a solution in the frequency domain for the problem of wave scattering by an infinite elastic thin plate with a cavity of arbitrary shape. The methods outlined in this paper have applications in ice sheet modelling, and the development of offshore floating airports as well as other marine structures. There are also applications to problems in electrostatics involving non-circular, finitely conducting inclusions, and how the bulk properties of composite plates are affected by cavities, particularly by arrays of cavities (Yardley et al. [162], and Norris and Vemula [111]). The formulation outlined in this paper can easily incorporate more complicated shapes and boundary conditions, requiring only that the boundary be smooth. Cavities with wedges or corner points encounter the additional difficulty of hypersingular integral equations, which is discussed by Stern [146] but is not considered in the present work.

The outline of this paper is as follows. In section 2 we formulate the problem in terms of a system of boundary integral equations (BIE), using the commutativity property of the biharmonic plate equation. In section 3 we consider the most challenging case of free-edge conditions at the cavity edge, and derive a solution by expanding the unknowns in terms of Fourier series. In section 4 we show how the CBIEM approach can be used to control irregular frequencies. In section 5
we give expressions for the solution outside the cavity, and section 6 extends the formulation to other edge conditions. In section 7 we consider the vibration of a plate of arbitrary geometry using our formulation, primarily as a check on our numerics. We give results in section 8, beginning with a comparison with results previously computed for the vibration frequencies by Singh and Chakraverty [132, 133, 134] using Rayleigh-Ritz methods. While these Rayleigh-Ritz methods, especially those based on finite elements, have proved very effective for determining the frequencies of vibration, there might be situations where methods based on integral equations are preferable (particularly if the geometry has a smooth boundary). Our results show generally good agreement with these previous estimates, which gives us a high level of confidence in the accuracy of our numerical solution. We then consider scattering by obstacles of various shapes. We give a brief summary in section 9.

5.3.2 Problem formulation

We begin with the non-dimensional elastic, thin-plate equation in the frequency domain given by

\[(\Delta^2 - k^4)w = (\Delta + k^2)(\Delta - k^2)w = 0,\]  

(R.1)

where free-edge, clamped or simply-supported boundary conditions are imposed at the cavity edge and \(k^2 = \omega\sqrt{\rho h/D}\), where \(\omega\) is the angular frequency, \(\rho\) is the mass density, \(h\) is the thickness and \(D\) is the flexural rigidity of the plate. We decompose the displacement of the plate \(w\) into incident and scattered components \((w = w^I + w^S)\) where \(w^S\) satisfies Sommerfeld’s radiation condition, and for simplicity, we take a propagating incident plane wave \(w^I = e^{ikr\cos\theta}\). As a matter of convention we let \(\Omega\) denote the smooth cavity in our infinite plate with boundary \(\partial\Omega\), and the region outside the cavity (occupied by the plate) is defined by \(\Omega_\infty\) which has a boundary at infinity denoted by \(\partial\Omega_\infty\) as shown in Figure 5.2.

If we define \(\Psi = (\Delta + k^2)w^S\) and \(\Phi = (\Delta - k^2)w^S\), it is clear that \(\Psi\) satisfies the anti-Helmholtz equation

\[(\Delta - k^2)\Phi = 0,\]  

(R.2a)

and \(\Phi\) satisfies the Helmholtz equation

\[(\Delta + k^2)\Phi = 0.\]  

(R.2b)

We define the Green’s identity

\[
\int_{\partial\Omega} \{ \phi(x') \partial_{n'} G^{A,H}(x, x') - \partial_n \phi(x') G^{A,H}(x, x') \} \ dS' = \begin{cases} 
\phi(x) & \text{if } x \in \Omega_\infty \\
\frac{1}{2} \phi(x) & \text{if } x \in \partial\Omega \\
0 & \text{if } x \in \Omega
\end{cases} \]  

(R.3)
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Figure 5.2: Infinite plate with a cavity of arbitrary shape.

where \( G^H \) and \( G^A \) denote the Green’s functions for the Helmholtz and anti-Helmholtz equations respectively (and \( \phi, G^{A,H} \) satisfy the same differential equation and smoothness conditions specified in Kitahara [62]). We define these as \( G^H(x,x') = \frac{1}{4}H_0^{(1)}(k|x-x'|) \) and \( G^A(x,x') = \frac{1}{4}H_0^{(1)}(ik|x-x'|) \), where \( H_0^{(1)} \) denotes a zero-order Hankel function of the first kind, with \( x \) and \( x' \) representing the field and source points respectively.

Applying Green’s identity to the Helmholtz and anti-Helmholtz expressions in (R.2a) and (R.2b) yields the system

\[
\frac{1}{2}\Psi = \int_{\partial\Omega} \left\{ \partial_{n'}G^A \Psi - \partial_{n'}\Psi G^A \right\} dS', \quad \text{(R.4a)}
\]

\[
\frac{1}{2}\Phi = \int_{\partial\Omega} \left\{ \partial_{n'}G^H \Phi - \partial_{n'}\Phi G^H \right\} dS', \quad \text{(R.4b)}
\]

where the integrals around the boundary at infinity \( \partial\Omega_\infty \) disappear due to Sommerfeld’s radiation condition. From this, our system can be expressed in terms of the scattered displacement directly

\[
\frac{1}{2}(\Delta + k^2)w^S = \int_{\partial\Omega} \left\{ \partial_{n'}G^A(\Delta + k^2)w^S - \partial_{n'}(\Delta + k^2)w^S G^A \right\} dS', \quad \text{(R.5a)}
\]

\[
\frac{1}{2}(\Delta - k^2)w^S = \int_{\partial\Omega} \left\{ \partial_{n'}G^H(\Delta - k^2)w^S - \partial_{n'}(\Delta - k^2)w^S G^H \right\} dS'. \quad \text{(R.5b)}
\]

After observing that \( w^S = w - w^I \) and

\[
(\Delta + k^2)w^I = 0, \quad \text{(R.6a)}
\]

\[
(\Delta - k^2)w^I = -2k^2w^I, \quad \text{(R.6b)}
\]
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Equations (R.5a) and (R.5b) can then be expressed in terms of the total displacement

\[
\frac{1}{2}(\Delta + k^2) w = \int_{\partial \Omega} \{ \partial_{n'} G^A(\Delta + k^2) w - \partial_{n'} (\Delta + k^2) w G^A \} \, dS', \quad \text{(R.7a)}
\]

\[
\frac{1}{2}(\Delta - k^2) w + k^2 w^I = \int_{\partial \Omega} \{ \partial_{n'} G^H(\Delta - k^2) w^S - \partial_{n'} (\Delta - k^2) w^S G^H \} \, dS' + 2k^2 \int_{\partial \Omega} \{ \partial_{n'} G^H w^I - \partial_{n'} w^I G^H \} \, dS'. \quad \text{(R.7b)}
\]

To evaluate the final integral in equation (R.7b), we apply Green’s identity to the incident displacement alone, which yields an expression involving an integral around the boundary at infinity

\[
\frac{1}{2} w^I = \int_{\partial \Omega} \{ \partial_{n'} G^H w^I - \partial_{n'} w^I G^H \} \, dS' + \int_{\partial \Omega_\infty} \{ \partial_{n'} G^H w^I - \partial_{n'} w^I G^H \} \, dS', \quad \text{(R.8)}
\]

where it can be shown from (R.3) that \( \int_{\partial \Omega_\infty} \{ \partial_{n'} G^H w^I - \partial_{n'} w^I G^H \} \, dS' = w^I \).

Consequently, the total plate displacement at the edge of the cavity can be expressed by the following system of boundary integral equations

\[
\frac{1}{2}(\Delta + k^2) w = \int_{\partial \Omega} \{ \partial_{n'} G^A(\Delta + k^2) w - \partial_{n'} (\Delta + k^2) w G^A \} \, dS', \quad \text{(R.9a)}
\]

\[
\frac{1}{2}(\Delta - k^2) w + 2k^2 w^I + \int_{\partial \Omega_\infty} \{ \partial_{n'} G^H(\Delta - k^2) w - \partial_{n'} (\Delta - k^2) w G^H \} \, dS'. \quad \text{(R.9b)}
\]

These equations will be the basis of our numerical solution for computing the plate displacement. This expression extends the boundary integral equation method for Helmholtz’s equation to the biharmonic plate equation.

5.3.3 Free-edge boundary conditions

The free-edge boundary conditions for a smooth cavity can be represented in the form

\[
\Delta w = \beta (\partial_s^2 + \partial_s \theta \partial_{n'}) \, w, \quad \text{(R.10a)}
\]

\[
\partial_{n'} \Delta w = \beta (\partial_s^2 \theta + \partial_s \theta \partial_s^2 - \partial_s^2 \partial_{n'}) \, w, \quad \text{(R.10b)}
\]

(Stern [146]) where \( \partial_s = \nabla \cdot s \), \( \partial_n = \nabla \cdot n \), \( s \) is a unit vector tangential to \( \partial \Omega \), \( n \) and \( n' \) denote unit normal vectors to \( \partial \Omega \) (with respect to the field and source points respectively), and \( \beta = 1 - \nu \), where \( \nu \) is the Poisson ratio of the plate. We introduce the notation \( \Theta = \Theta(s) \) for the direction cosine such that \( n = (\cos \Theta, \sin \Theta) \). We also assume that the cavity boundary \( \partial \Omega \) is parametrised by \( s \) such that \( s \in (-\gamma, \gamma) \), using the notation \( s \) and \( s' \) when referring to the field and source points respectively.

In a manner identical to Bennetts and Williams [12], for the case of free-edge conditions, we
expand the resulting two unknowns (the displacement and the normal derivative of the displacement) in equations (R.9a) and (R.9b) in terms of the following Fourier series

\[ w = \sum_{m=-\infty}^{\infty} c_m e^{im\pi s'/\gamma}, \quad \partial_{n'} w = \sum_{m=-\infty}^{\infty} d_m e^{im\pi s'/\gamma}. \]  

(R.11)

and then apply the relevant boundary conditions.

Implementing the Fourier Series above allows equations (R.9a) and (R.9b) to take the form

\[
\sum_{m=-\infty}^{\infty} \left\{ \frac{1}{2} \left[ k^2 - \beta (m\pi/\gamma)^2 \right] e^{im\pi s'/\gamma} \right\} c_m + \left\{ \frac{1}{2} \beta \partial_s \Theta(s) e^{im\pi s'/\gamma} \right\} d_m = \\
\sum_{m=-\infty}^{\infty} c_m \left\{ \left[ k^2 - \beta (m\pi/\gamma)^2 \right] \int_{\partial\Omega} \partial_{n'} G^L e^{im\pi s'/\gamma} dS' \right. \\
+ \beta (m\pi/\gamma)^2 \int_{\partial\Omega} \partial_s \Theta(s') G^L e^{im\pi s'/\gamma} dS' - \int_{\partial\Omega} \partial_{s'} \Theta(s') G^L e^{im\pi s'/\gamma} dS' \bigg\} \\
+ d_m \left\{ \beta \int_{\partial\Omega} \partial_s \Theta(s') \partial_{n'} G^L e^{im\pi s'/\gamma} dS' - \left[ k^2 + \beta (m\pi/\gamma)^2 \right] \int_{\partial\Omega} G^L e^{im\pi s'/\gamma} dS' \right\}, \tag{R.12a}
\]

and

\[
\sum_{m=-\infty}^{\infty} \left\{ \frac{1}{2} \beta \partial_s \Theta(s) e^{im\pi s'/\gamma} \right\} d_m - \left\{ \frac{1}{2} \left[ k^2 + \beta (m\pi/\gamma)^2 \right] e^{im\pi s'/\gamma} \right\} c_m = \\
-2k^2 w^{1} + \sum_{m=-\infty}^{\infty} c_m \left\{ - \left[ k^2 + \beta (m\pi/\gamma)^2 \right] \int_{\partial\Omega} \partial_{n'} G^R e^{im\pi s'/\gamma} dS' \right. \\
+ \beta (m\pi/\gamma)^2 \int_{\partial\Omega} \partial_s \Theta(s') G^R e^{im\pi s'/\gamma} dS' - \int_{\partial\Omega} \partial_{s'} \Theta(s') G^R e^{im\pi s'/\gamma} dS' \bigg\} \\
+ d_m \left\{ \beta \int_{\partial\Omega} \partial_s \Theta(s') \partial_{n'} G^R e^{im\pi s'/\gamma} dS' + \left[ k^2 - \beta (m\pi/\gamma)^2 \right] \int_{\partial\Omega} G^R e^{im\pi s'/\gamma} dS' \right\}. \tag{R.12b}
\]

Multiplying these resulting expressions by \(e^{im\pi s'/\gamma}\), truncating the Fourier series, and integrating over the cavity boundary \(\partial\Omega\) with respect to the field point yields the block-matrix system:

\[
Mz = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} z = f, \tag{R.13}
\]
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with the component blocks of $M$ defined as

$$
M_{11} = \left[ \frac{1}{2} (k^2 I - \eta N^2) \right] G + \xi NB^A - \eta N^2 A^A - \left( k^2 I - \eta N^2 \right) E^A,
$$

(R.14a)

$$
M_{12} = \left[ \frac{1}{2} \beta H - \beta C^A + \left( k^2 I + \eta N^2 \right) D^A \right]
$$

(R.14b)

$$
M_{21} = \left[ \left( k^2 I + \eta N^2 \right) E^H + \xi NB^H - \eta N^2 A^H - \frac{1}{2} \left( k^2 I + \eta N^2 \right) G \right]
$$

(R.14c)

$$
M_{22} = \left[ \frac{1}{2} \beta H - \beta C^H - \left( k^2 I - \eta N^2 \right) D^H \right],
$$

(R.14d)

where $I$ denotes the identity matrix, $z = [c, c, \ldots, d_M, \ldots, d_{-M}]^T$, $f = [0, g]^T$, $\xi = i \beta \pi / \gamma$, $\eta = \beta \pi^2 / \gamma^2$.

$$
A_{mn}^{A,H} = \int_\partial \int_\partial \Theta(s') G^{A,H} e^{im\pi s' / \gamma} e^{i\pi s / \gamma} dS' dS,
$$

$$
G_{mn} = \int_\partial e^{i(m+n)\pi s / \gamma} dS,
$$

$$
H_{mn} = \int_\partial \partial_s \Theta(s) e^{i(m+n)\pi s / \gamma} dS,
$$

$$
N_{mn} = n\delta_{mn},
$$

$$
g_0 = -2k^2 \int_\partial W e^{i\pi s / \gamma} dS,
$$

Note that in the definitions above, if the superscript $A$ is used, then the anti-Helmholtz Green’s function $G^A$ is used to compute the matrix. Similarly, if the superscript $H$ is used, then the Helmholtz Green’s function $G^H$ is used, for example,

$$
A_{mn}^A = \int_\partial \int_\partial \partial_s \Theta(s') G^A e^{im\pi s' / \gamma} e^{i\pi s / \gamma} dS' dS.
$$

As is the case for Helmholtz’s equation, for certain frequencies $k$ the matrix $M$ is non-invertible, giving rise to problems with ill-conditioning. The control of this phenomenon is the subject of the following section.

5.3.4 Combined boundary integral equation method

In order to numerically control ill-conditioning we apply the technique as discussed by Lau and Hearn [72], which is known as the combined boundary integral equation method (CBIEM). This technique involves overdetermining our existing system of equations by positioning the field point inside the cavity $\Omega$, with each point (denoted by $p_j$) giving rise to an additional set of
The positions of these points is important as they must not coincide with the nodal points of interior eigenfunctions, and these are unknown in general (Lau and Hearn [72]). To overcome this, we sample our field points from a ball of small (but not too small) radius centred about the origin. Another ill-conditioned problem is encountered if the ball is too small, as only the zero-order Bessel functions will contribute towards controlling ill-conditioning. Likewise, if the radius is too big and we sample field points that are too close to the boundary, then we encounter problems in connection with the Green’s identity (R.3).

The locations of these resonant wavenumbers are also unknown a priori in general, and correspond to the eigenvalues of an interior problem whose boundary conditions are unknown, with the added difficulty that these adjoint conditions do not relate to a physical problem. Unfortunately, there is no simple connection as in Helmholtz’s equation, where an external Dirichlet problem has irregular frequencies corresponding to the eigensolutions of the corresponding internal Neumann problem, and vice-versa (Kleinman and Roach [64]).

Implementing the free-edge boundary conditions and truncating the Fourier series expansions (R.11), in an approach similar to that above, gives rise to the CBIEM system for the point \( \hat{p}_j \)

\[
\hat{M}_j z = \hat{f}_j,
\]  

where

\[
\hat{M}_j = \left[ \frac{(k^2I - \eta N^2) \hat{H}^j + \eta N^2 \hat{A}^j - \xi N \hat{B}^j}{\eta N^2D^j - \xi N E^j - (k^2I + \eta N^2)L^j} \beta \hat{C}^j - (k^2I + \eta N^2) \hat{J}^j \right].
\]  

with \( \hat{f} = [0, \hat{g}]^T \), \( \hat{g} = 2k^2w_1(\hat{p}_j) \), and

\[
\hat{A}_m^j = \int_{\Omega} \partial_{s'} \Theta(s') G^A(\hat{p}_j, x') e^{i\pi s'/\gamma} dS',
\]

\[
\hat{B}_m^j = \int_{\Omega} \partial_{s'} \Theta(s') G^A(\hat{p}_j, x') e^{i\pi s'/\gamma} dS',
\]

\[
\hat{C}_m^j = \int_{\Omega} \partial_{s'} \Theta(s') \partial_{s''} G^A(\hat{p}_j, x') e^{i\pi s'/\gamma} dS',
\]

\[
\hat{D}_m^j = \int_{\Omega} \partial_{s'} \Theta(s') G^H(\hat{p}_j, x') e^{i\pi s'/\gamma} dS',
\]

\[
\hat{E}_m^j = \int_{\Omega} \partial_{s'} \Theta(s') G^H(\hat{p}_j, x') e^{i\pi s'/\gamma} dS'.
\]
Incorporating the CBIEM system(s) with our standard system gives the overdetermined problem

\[
\tilde{M}z = \begin{bmatrix} M \\ \tilde{M}_1 \\ \tilde{M}_2 \\ \vdots \end{bmatrix} z = \begin{bmatrix} f \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \end{bmatrix} = \tilde{f},
\]  

(R.18)

which can be solved by using a standard Moore–Penrose generalised inverse. This solution controls irregular frequency effects, provided that \( \tilde{f} \in \text{col}(\tilde{M}) \).

### 5.3.5 Solution outside of cavity boundary

Using the solution for the two unknowns at the cavity edge \( \partial \Omega \), the displacement of the plate for a point in the exterior region \( \Omega_\infty \) can be computed. This is achieved in general by combining equations (R.9a) and (R.9b) to obtain

\[
w(x) = w^1(x) - \int_{\partial \Omega} \left\{ \partial_{n^\prime} G^P(x, x^\prime)(\Delta - k^2)w(x^\prime) - \partial_{n^\prime}(\Delta - k^2)w(x^\prime)G^P(x, x^\prime) \right\} dS^\prime \\
+ \int_{\partial \Omega} \left\{ \partial_{n^\prime} G^A(x, x^\prime)w(x^\prime) - \partial_{n^\prime}w(x^\prime)G^A(x, x^\prime) \right\} dS^\prime,
\]  

(R.19)

where

\[
G^P(x, x^\prime) = \frac{1}{2k^2} \left[ G^H(x, x^\prime) - G^A(x, x^\prime) \right] = \frac{i}{8k^2} \left[ H_0^{(1)}(k|x - x^\prime|) - H_0^{(1)}(ik|x - x^\prime|) \right],
\]  

(R.20)

is the Green’s function which satisfies the non-dimensional plate equation given by

\[
(\Delta^2 - k^4)G^P(x, x^\prime) = \delta(x - x^\prime).
\]  

(Norris and Vemula [111]). Substituting the boundary conditions (R.10a) and (R.10b) into expression (R.19) allows us to compute the displacement of the plate in \( \Omega_\infty \) for the case of a free-edge cavity

\[
w(x) = w^1(x) - \int_{\partial \Omega} \left\{ \partial_{n^\prime} G^P(x, x^\prime) \left[ \beta \partial_{n^\prime}^2 - k^2 \right] w(x^\prime) \\
- \beta \left[ \partial_{n^\prime} \Theta \partial_{n^\prime} + \partial_{n^\prime} \Theta \partial_{n^\prime}^2 \right] w(x^\prime)G^P(x, x^\prime) \right\} dS^\prime \\
- \int_{\partial \Omega} \left\{ \partial_{n^\prime} G^P(x, x^\prime) \left[ \beta \partial_{n^\prime} \Theta \right] \partial_{n^\prime}w(x^\prime) + \left[ \beta \partial_{n^\prime}^2 + k^2 \right] \partial_{n^\prime}w(x^\prime)G^P(x, x^\prime) \right\} dS^\prime \\
+ \int_{\partial \Omega} \left\{ \partial_{n^\prime} G^A(x, x^\prime)w(x^\prime) - \partial_{n^\prime}w(x^\prime)G^A(x, x^\prime) \right\} dS^\prime.
\]  

(R.22)
5.3.6 Other boundary conditions

One advantage of formulating this problem in terms of boundary integral equations is that different boundary conditions can be easily considered and implemented (although the free-edge case is the most challenging). This includes clamped boundary conditions given by

\[ w = 0, \quad \partial_{n'} w = 0. \quad \text{(R.23)} \]

Applying these conditions to equations (R.9a) and (R.9b), and expanding \( \Delta w = \sum c_m e^{im\pi s'/\gamma} \), and \( \partial_{n'} \Delta w = \sum d_m e^{im\pi s'/\gamma} \) admits a system of equations similar to expressions (R.13) and (R.16) except that

\[
M = \begin{bmatrix}
\frac{1}{2} G - E^A & D^A \\
\frac{1}{2} G - E^H & D^H
\end{bmatrix}, \quad \tilde{M}_j = \begin{bmatrix}
\tilde{H} & -\tilde{J} \\
\tilde{L} & -K
\end{bmatrix}.
\quad \text{(R.24)}
\]

Note that for clamped boundary conditions, a Fourier series expansion for the unknowns \( \Delta w \) and \( \partial_{n'} \Delta w \) is not actually required (as there are no tangential derivative terms present in the boundary conditions), but it is introduced to clarify the equations.

The plate displacement in \( \Omega_\infty \) is computed in a manner similar to the free-edge case, and it is given by

\[
w(x) = w^I(x) - \int_{\partial \Omega} \left\{ \beta \partial_s \Theta \partial_{n'} w(x') - \partial_{n'} \Delta w(x') G^P(x, x') \right\} dS'. \quad \text{(R.25)}
\]

For the case of a smooth boundary, applying the simply-supported boundary conditions given by

\[ w = 0, \quad \Delta w - \beta \partial_s \Theta \partial_{n'} w = 0, \quad \text{(R.26)} \]

in a manner similar to before, it can be shown that

\[
M = \begin{bmatrix}
D^A & \frac{1}{2} \beta H - \beta C^A + k^2 D^A \\
\frac{1}{2} \beta H - \beta C^H - k^2 D^H
\end{bmatrix}, \quad \tilde{M}_j = \begin{bmatrix}
-\tilde{J} & \beta \tilde{C}^j - k^2 \tilde{J} \\
-K & \beta \tilde{G}^j + k^2 \tilde{K}^j
\end{bmatrix},
\quad \text{(R.27)}
\]

where \( \partial_{n'} \Delta w = \sum c_m e^{im\pi s'/\gamma} \), and \( \partial_{n'} w = \sum d_m e^{im\pi s'/\gamma} \), with the plate displacement in \( \Omega_\infty \) given by

\[
w(x) = w^I(x) - \int_{\partial \Omega} \left\{ \left[ \beta \partial_s \Theta \partial_{n'} G^P(x, x') + G^A(x, x') + k^2 G^P \right] \partial_{n'} w(x') \right\} dS'
\]

\[
+ \int_{\partial \Omega} \left\{ G^P(x, x') \partial_{n'} \Delta w(x') \right\} dS'. \quad \text{(R.28)}
\]
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5.3.7 Arbitrary finite plate vibration problem

Previously we considered the problem of wave scattering in an infinite plate with a closed cavity. A connection is now established between the solution method for this problem and the vibration problem of a finite plate surrounded by a vacuum. This second problem is shown in Figure 5.3.

Formulating the finite plate vibration problem in terms of a system of boundary integral equations yields a system identical to equations (R.9a) and (R.9b), except that there is no incident displacement term present. However, in order to relate this system in terms of the vibration problem, we need to reorient the normal derivative and look closely at single-layer distribution terms present in our integral equations, taking note that there is a jump discontinuity at the boundary [65] given by

\[
\lim_{x \to x^-} \int_{\partial \Omega} w(x') \partial_n' G(x, x') \mathrm{d}S' = w(x) + \int_{\partial \Omega} w(x') \partial_n G(x, x') \mathrm{d}S',
\]

(R.29)

where \( x \to x^- \) denotes the field point approaching the boundary from the interior region \( \Omega \). Note that the normal derivative in the final integral of equation (R.29) must be suitably oriented.

Consequently, the boundary integral system for computing the vibration of a finite plate is given in general by

\[
-\frac{1}{2}(\Delta + k^2)w = \int_{\partial \Omega} \left\{ \partial_n G^A(\Delta + k^2)w + \partial_n(\Delta + k^2)w G^A \right\} \mathrm{d}S',
\]

(R.30a)

\[
-\frac{1}{2}(\Delta - k^2)w = \int_{\partial \Omega} \left\{ \partial_n G^H(\Delta - k^2)w + \partial_n(\Delta - k^2)w G^H \right\} \mathrm{d}S'.
\]

(R.30b)

If we implement the reoriented free-edge boundary conditions given by

\[
\Delta w = \beta \left( \partial_s^2 - \partial_s \partial_n \right) w,
\]

(R.31a)

\[
\partial_n' \Delta w = -\beta \left( \partial_s^2 \partial_n + \partial_s \partial_s^2 + \partial_s^2 \partial_n' \right) w,
\]

(R.31b)

use the series expansion given by equation (R.11), multiply by \( e^{i\pi s/\gamma} \) and integrate over the cavity edge \( \partial \Omega \) with respect to the field point, we obtain the following block-matrix system

\[
P(k)z = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0,
\]

(R.32)
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Figure 5.3: Vibration of an arbitrarily shaped smooth plate.

The vibration frequencies of an arbitrary plate are denoted by \( \mu_l = k_l^2 \), where \( k_l \) corresponds to the values of \( k \) such that \( P(k) \) has zero eigenvalue. It is important to note that these eigenvalues do not correspond to the irregular frequencies for the external cavity problem with free-edge, simply-supported or clamped boundary conditions. That is, the irregular frequencies do not match numerically with the vibration frequencies, and they do not correspond to a physical problem, Burton and Miller [21]. If these values were known, then techniques to control ill-conditioning would only need to be applied in the neighbourhood of each irregular frequency.

For clamped boundary conditions (which are unchanged when considering the vibration problem) we obtain a system similar to (R.32), except that

\[
P(k) = \begin{bmatrix} \frac{1}{2} G + E^A & D^A \\ \frac{1}{2} G + E^H & D^H \end{bmatrix}.
\]

(R.34)

For simply-supported boundary conditions, after adjusting the normal derivative for the vibration problem, we obtain

\[
w = 0, \quad \Delta w = -\beta \partial_n \Theta \partial_n w,
\]

(R.35)

which admits the system in (R.32), where

\[
P(k) = \begin{bmatrix} -D^A & \frac{1}{2} \beta H - k^2 D^A + \beta C^A \\ -D^H & \frac{1}{2} \beta H + k^2 D^H + \beta C^H \end{bmatrix}.
\]

(R.36)
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5.3.8 Computing the frequencies of vibration

The standard method used to determine vibration frequencies is finite-element methods (FEM), however, in this paper values are computed using a first order expansion of $P(k)$, giving rise to a generalised eigenvalue problem for the update $\kappa$ based on an initial estimate $k_0$ as follows:

$$P(k_0)z = -\kappa P'(k_0)z, \quad \text{where } \kappa = k - k_0,$$

where the update is given by the real part of the minimum eigenvalue $|\kappa|$ for the system (as $k$ is real).

If we compute the derivative of $P(k_0)$ using a finite difference method we can obtain the expression

$$P'(k_0) = \frac{1}{2\Delta k_0} [P(k_0 + \Delta k_0) - P(k_0 - \Delta k_0)],$$

where $\Delta k_0$ denotes a small change in $k_0$. However if $\Delta k_0$ is too small, then the elements of $P'(k_0)$ become large (due to the $1/2\Delta k_0$ term), giving rise to an ill-conditioned matrix. In such a case, this method becomes highly sensitive to the initial estimate for the vibration frequency, with convergence issues arising unless the initial estimate is suitably close to the real root. In general, accuracy for this method may only be valid to a few decimal places. This method is proposed primarily to confirm the results obtained by Singh and Chakraverty [132, 133, 134], using their values as a starting point for the algorithm. Alternatively one could search through $k$ space for values where $\det P = 0$, however the determinant of this matrix is large and isolating the roots can be difficult, especially in regions where the matrix $P$ is ill-conditioned. Nevertheless, calculation of these vibration frequencies acts as a very useful check of our numerical code.

5.3.9 Computing the displacement of the plate

Having solved the eigenvalue problem given by equation (R.32) we take the eigenvector of $P(k)$ that corresponds to the eigenvalue numerically closest to zero. This eigenvector corresponds to a vector of coefficients $z$, which is then used to compute the two appropriate unknowns at the plate edge by using the series expansions given in (R.11).

The expression for the plate displacement here is found in a manner identical to that for the cavity problem. By combining equations (R.30a) and (R.30b) we can achieve in general

$$w(x) = 2 \int_{\partial \Omega} \left\{ \partial_{x'} G^P(x, x') \Delta w(x') + \partial_{w'} \Delta w(x') G^P(x, x') \right\} dS'$$

$$- \int_{\partial \Omega} \left\{ \left[ \partial_{w'} G^A(x, x') + \partial_{\nu'} G^H(x, x') \right] w(x') + \partial_{w'} w(x') \left[ G^A(x, x') + G^H(x, x') \right] \right\} dS', \quad (R.39)$$
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for the displacement of the plate.

Substituting the boundary conditions (R.31a) and (R.31b) into expression (R.39) allows us to compute the displacement of the plate for the case of free-edge boundary conditions

\[
w(x) = 2\beta \int_{\partial\Omega} \left\{ \left[ \partial_{n'}G^P(x, x') \partial_s^2 - G^P(x, x') \left( \partial_s^2 \Theta + \partial_s \partial_{s'} \Theta \partial_{s'}^2 \right) \right] w(x') \right. \\
\left. - \left[ \partial_{n'}G^P(x, x') \partial_s \Theta + G^P(x, x') \partial_{s'}^2 \right] \partial_{n'}w(x) \right\} dS'
\]

\[
- \int_{\partial\Omega} \left\{ \left[ \partial_{n'}G^A(x, x') + \partial_{n'}G^H(x, x') \right] w(x') \right. \\
\left. + \partial_{n'}w(x') \left[ G^A(x, x') + G^H(x, x') \right] \right\} dS'.
\]

Similarly for clamped boundary conditions which are given by (R.35) we obtain

\[
w(x) = 2 \int_{\partial\Omega} \left\{ \partial_{n'}G^P(x, x') \Delta w(x') + \partial_{n'}\Delta w(x')G^P(x, x') \right\} dS',
\]

and for simply-supported boundary conditions, which are given by (R.23), it can be shown that

\[
w(x) = 2 \int_{\partial\Omega} \left\{ G^P(x, x') \partial_{n'}\Delta w(x') - \left[ \beta \partial_s \Theta \partial_{n'}G^P(x, x') \right] \partial_{n'}w(x') \right\} dS'
\]

\[
- \int_{\partial\Omega} \left\{ \partial_{n'}w(x') \left[ G^A(x, x') + G^H(x, x') \right] \right\} dS'.
\]

5.3.10 Results

We begin by considering finite plates of circular and elliptical geometry. The circular finite plate acts as a check to validate numerical implementations since the vibration frequencies can be calculated straightforwardly, as specified in Leissa [73]. The following tables of vibration frequencies (\(\mu = k^2\)) were computed and compared with some published values. These values were generated using 1000 panels around the boundary and truncating the sums in (R.11) from \(-M\) to \(M\), with \(M = 10 - 20\) (depending on the degree of eccentricity of each ellipse and the relevant boundary condition). The values \(a\) and \(b\) denote the semi-major and semi-minor axes of an ellipse, with \(a = 1\) unless otherwise specified. We define \(\nu\) as the Poisson ratio.

The values from previous authors in the following tables are the lowest vibration frequencies corresponding to the first few modes of vibration (Singh and Chakraverty [132, 133, 134]). Accordingly for the case of a circular geometry we have chosen the values from Leissa [73] and Leissa and Narita [74] in a similar manner, although further frequency values are available in these publications.

Using the vibration frequency values from these tables, the following shapes for the displacement
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<table>
<thead>
<tr>
<th>Results</th>
<th>Frequency values</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>5.253</td>
<td>9.052</td>
<td>12.222</td>
</tr>
<tr>
<td>Clamped</td>
<td>Leissa [73]</td>
<td>10.216</td>
<td>21.260</td>
<td>34.880</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>10.205</td>
<td>21.235</td>
<td>34.832</td>
</tr>
<tr>
<td>Simply-supported†</td>
<td>Leissa and Narita [74]</td>
<td>4.953</td>
<td>13.898</td>
<td>25.613</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>4.948</td>
<td>13.933</td>
<td>25.676</td>
</tr>
</tbody>
</table>

Table 5.1: Lowest four vibration frequencies for a circular plate of radius $a = 1$ ($\nu = 0.33$, †: $\nu = 0.30$).

<table>
<thead>
<tr>
<th>Results</th>
<th>$b/a$</th>
<th>Frequency values</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>6.056</td>
<td>6.570</td>
<td>12.112</td>
<td>14.733</td>
</tr>
<tr>
<td>Singh and Chakraverty [132]</td>
<td>0.6</td>
<td>6.492</td>
<td>8.680</td>
<td>16.302</td>
<td>18.765</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>6.472</td>
<td>8.641</td>
<td>16.242</td>
<td>18.714</td>
</tr>
<tr>
<td>Singh and Chakraverty [132]</td>
<td>0.4</td>
<td>6.662</td>
<td>12.827</td>
<td>17.037</td>
<td>25.946</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>6.642</td>
<td>12.848</td>
<td>17.014</td>
<td>25.944</td>
</tr>
</tbody>
</table>

Table 5.2: Lowest four vibration frequencies for a free elliptic plate ($\nu = 0.33$).

of the plate are generated in Figures 5.12 and 5.13, for the case of a clamped circular plate and a free elliptic plate ($b/a = 0.8$). These figures were generated using 400 panels at the plate boundary with $M = 10$. The mode shapes generated in Figure 5.12 for the circular plate are in keeping with the shapes that would be obtained by separation of variables, and the elliptical mode shapes in Figure 5.13 are similar to the modes for a free-edge circular plate (but distorted) and also bear a strong resemblance to the shapes obtained in Singh and Chakraverty [134]. In Figure 5.12(d) there is a small amount of error at the plate edge, which appears to arise from dipolar term effects.

We now consider scattering by a finite closed cavity in an infinite plate. Firstly, to demonstrate the effectiveness of the combined boundary element method, the maximum absolute values of $\Delta w$, $w$ and $\partial_n \Delta w$ (for clamped, free-edge and simply-supported boundary conditions, respectively) were plotted for varying $k$, using both the standard boundary integral equation approach and the combined boundary integral equation method.

From figures 5.4(a) to 5.4(c), we can see that the CBIEM approach is able to efficiently control ill-conditioning when recovering unknowns at the cavity edge, for a range of frequency values, particularly at the irregular frequencies located at $k \approx 1.89, 3.17, 4.57, 5.00, 6.02$ and $6.12$ (note that the first irregular frequency is less pronounced for the clamped and simply-supported cases). These Figures were generated for an elliptical cavity ($b/a = 2, a = 1$) with $M = 50$ solution modes, 500 panels, 30 CBIEM points and an incident potential $w^i = J_0(kr)$.

The scattering displacement fields for circular, elliptical, $L_4$ and star cavities were generated
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<table>
<thead>
<tr>
<th>Results</th>
<th>$b/a$</th>
<th>Frequency values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singh and Chakraverty [133]</td>
<td>0.8</td>
<td>6.394 15.634 20.012 29.139</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>6.391 15.636 20.022 29.137</td>
</tr>
<tr>
<td>Singh and Chakraverty [133]</td>
<td>0.6</td>
<td>9.763 19.566 33.122 33.777</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>9.759 19.564 33.107 33.774</td>
</tr>
<tr>
<td>Singh and Chakraverty [133]</td>
<td>0.4</td>
<td>19.514 31.146 46.823 66.937</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>19.494 31.192 46.787 66.921</td>
</tr>
</tbody>
</table>

Table 5.3: Lowest four vibration frequencies for a simply-supported elliptic plate ($\nu = 0.3$).

<table>
<thead>
<tr>
<th>Results</th>
<th>$b/a$</th>
<th>Frequency values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singh and Chakraverty [134]</td>
<td>0.8</td>
<td>13.250 51.570 115.500</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>13.281 51.738 115.376</td>
</tr>
<tr>
<td>Singh and Chakraverty [134]</td>
<td>0.6</td>
<td>20.340 79.170 177.400</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>20.323 79.311 176.971</td>
</tr>
<tr>
<td>Singh and Chakraverty [134]</td>
<td>0.4</td>
<td>41.600 162.000 362.900</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>41.801 161.461 364.890</td>
</tr>
</tbody>
</table>

Table 5.4: Lowest three vibration frequencies for a clamped elliptic plate.

for free-edge, clamped and simply-supported boundary conditions in Figures 5.6 to 5.11, for an incident plane wave displacement incoming from the left. Corresponding line plots at $y = 0$ are also shown. The cavities under consideration are shown in Figure 5.5. Note that the $L_4$ cavity is called that due to its connection with the $L_4$ norm (for example, $x^4 + y^4 = 1$).

Primarily, the most interesting behaviour is exhibited by the star cavity which features back-scattering and resolution-blur for the clamped and simply-supported edge conditions in Figures 5.7 and 5.11. In Figure 5.9 we see channeling for the free-edge $L_4$ and star cavities, which originate from the corners of the cavities furthest to the right (in the direction of the incident wave propagation). It would appear that for all 3 boundary conditions, the scattering profile for the $L_4$ cavity qualitatively appears to be that of a stretched circular profile. There is no such relationship between the circular and elliptical cavities, with strong differences in the scattered displacement fields close to the point where the incident wave first strikes the cavity.

Our solution for the circular cavity shows strong agreement with the solution generated by the method outlined in Appendix 5.3.12, for the three types of boundary conditions considered. The scattered displacement fields were generated using values $k = 1$, $M = 50$, 10 incident modes, 500 panels, 15 CBIEM points, and $\beta = 1/2$ where applicable, with the area of the cavities preserved at a constant value of 10. For example, the elliptical cavity has semi-major axes $a = 1.2615$ and $b = 2.5231$ so that the area $\pi a b \approx 10$ (and $b/a = 2$).
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5.3.11 Conclusions

In this paper we have demonstrated how to compute the displacement of an infinite plate, with a single arbitrarily-shaped cavity about the origin, subject to an incident wave forcing in the frequency domain. Specifically, we have presented solutions for 3 types of boundary conditions, and have numerically demonstrated these with 4 geometries. It would be straightforward to implement more complicated geometries and other boundary conditions. We have also shown that by formulating this problem in terms of boundary integral equations, we encounter difficulties with irregular frequencies, which are controlled effectively using the combined boundary element method.

We have shown how to formulate the finite plate vibration problem as a system of boundary integral equations, and have computed the first few eigenvalues for specified circular and elliptical plates. Additionally we have computed the modes of vibration for two geometries with two different boundary conditions. The results from these computations acts as validation of our numerical solution.
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### Figure 5.5: Cavity geometries considered for computing plate displacement in $\Omega_\infty$: (a) Circular cavity; (b) Elliptical cavity; (c) $L_4$ cavity; and (d) Star cavity.

An algorithm to compute the eigenvalues is also discussed, based on a first order expansion of the system, with results that generally agree well with literature values. In conclusion, the method we have described and verified will prove useful in a range of problems concerned with single cavities and inclusions. Of particular interest will be its application to arrays of cavities, which have already been shown by Movchan et al. [106] to have very interesting band-gap properties.

### 5.3.12 Appendix A: Alternative approach for circular cavities

The displacement of the plate in $\Omega_\infty$ can be computed simply for the case of a circular cavity. This is outlined by Konenkov [67], Norris and Vemula [111], who expand the solution for a circle of radius $a$ as follows

$$w = w^I + \sum_{n=-\infty}^{\infty} \left[ A_n H_n^{(1)}(kr) + B_n H_n^{(1)}(ikr) \right] e^{in\theta}, \quad (R.43)$$

where $H_n^{(1)}$ are Hankel functions of the first kind. A standard incident plane wave is then considered, and subsequently represented in terms of cylindrical waves via the Jacobi–Anger identity

$$w^I = e^{ikr \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{in\theta}, \quad (R.44)$$
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Figure 5.6: Scattered displacement field (real component) and line plot at $y = 0$ under clamped boundary conditions for: (a)–(b) circular cavity; and (c)–(d) elliptical cavity.

Cakoni and Colton [22]. After applying the relevant boundary conditions at $r = a$, it is straightforward to obtain a system for the unknown coefficients in equation (R.43). We demonstrate this for the case of free-edge boundary conditions, given by

\[ \Delta w = \frac{\beta}{r} \left( \partial_r + \frac{1}{r^2} \partial^2_\theta \right) w, \]  
\[ \partial_r \Delta w = \frac{\beta}{r^2} \left( \frac{1}{r} - \partial_r \right) \partial^2_\theta w. \]  

(R.45a)  
(R.45b)

Applying these conditions at $r = a$ yields

\[
\begin{bmatrix}
  a_n & b_n \\
  c_n & d_n
\end{bmatrix}
\begin{bmatrix}
  A_n \\
  B_n
\end{bmatrix} = -i^n \begin{bmatrix}
  \epsilon_n \\
  f_n
\end{bmatrix},
\]  

(R.46)
Figure 5.7: Scattered displacement field (real component) and line plot at \( y = 0 \) under clamped boundary conditions for: (a)–(b) \( L_4 \) cavity; and (c)–(d) star cavity.

where

\[
a_n = \left[ \frac{\beta n^2}{a^2} - k^2 \right] H_n^{(1)}(ka) - \frac{\beta k}{2a} \left[ H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka) \right], \quad (R.47a)
\]
\[
b_n = \left[ \frac{\beta n^2}{a^2} + k^2 \right] H_n^{(1)}(ika) - \frac{i\beta k}{2a} \left[ H_{n-1}^{(1)}(ika) - H_{n+1}^{(1)}(ika) \right], \quad (R.47b)
\]
\[
c_n = -\frac{k}{2} \left[ \frac{\beta n^2}{a^2} + k^2 \right] \left[ H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka) \right] + \frac{\beta n^2}{a^3} H_n^{(1)}(ka), \quad (R.47c)
\]
\[
d_n = -\frac{ik}{2} \left[ \frac{\beta n^2}{a^2} - k^2 \right] \left[ H_{n-1}^{(1)}(ika) - H_{n+1}^{(1)}(ika) \right] + \frac{\beta n^2}{a^3} H_n^{(1)}(ika), \quad (R.47d)
\]
\[
e_n = \left[ \frac{\beta n^2}{a^2} - k^2 \right] J_n(ka) - \frac{\beta k}{2a} \left[ J_{n-1}(ka) - J_{n+1}(ka) \right], \quad (R.47e)
\]
\[
f_n = -\frac{k}{2} \left[ \frac{\beta n^2}{a^2} + k^2 \right] \left[ J_{n-1}(ka) - J_{n+1}(ka) \right] + \frac{\beta n^2}{a^3} J_n(ka). \quad (R.47f)
\]
Figure 5.8: Scattered displacement field (real component) and line plot at \( y = 0 \) under free-edge boundary conditions for: (a)–(b) circular cavity; and (c)–(d) elliptical cavity.

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5.3 Publication: “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems”

Figure 5.9: Scattered displacement field (real component) and line plot at $y = 0$ under free-edge boundary conditions for: (a)–(b) $L_4$ cavity; and (c)–(d) star cavity.
5.3 Publication: “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems”

Figure 5.10: Scattered displacement field (real component) and line plot at $y = 0$ under simply-supported boundary conditions for: (a)–(b) circular cavity; and (c)–(d) elliptical cavity.
5.3 Publication: “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems”

Figure 5.11: Scattered displacement field (real component) and line plot at $y = 0$ under simply-supported boundary conditions for: (a)–(b) $L_4$ cavity; and (c)–(d) star cavity.
5.3 Publication: “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems”

Figure 5.12: Plate displacement and associated contour plots for a clamped circular plate at: (a)–(b) \( \mu = 10.205 \); (c)–(d) \( \mu = 21.235 \); (e)–(f) \( \mu = 34.832 \); (g)–(h) \( \mu = 50.960 \).
Figure 5.13: Plate displacement and associated contour plots for a free elliptic plate ($b/a = 0.8$) at:
(a)–(b) $\mu = 6.056$; (c)–(d) $\mu = 6.570$; (e)–(f) $\mu = 12.112$; (g)–(h) $\mu = 14.733$. 
5.4 Arrays of arbitrarily shaped scatterers in one and two dimensions

In this section we consider one- and two-dimensional arrays of arbitrarily shaped scatterers. Previously we have examined a single arbitrary scatterer, as well as arrays of circular scatterers and pinned points. We now show that the solution procedure for a single arbitrary scatterer can be extended to the problem of a one-dimensional array by using Green’s functions which incorporate information about the grating. These Green’s functions are expressed in terms of multipoles, and require acceleration for effective numerical evaluation. Using the one-dimensional array solution, one can then extend the solution to a two-dimensional array problem using scattering matrices. We are interested here in altering the geometry of the inclusion in order to investigate the effect on the curvature of the band surfaces. This is relevant as the relative curvature of the band surfaces gives rise to interesting diffraction behaviours such as negative refraction.

In the paper “Flexural wave filtering and platonic polarisers in thin elastic plates” we examine gratings and two-dimensional square arrays of arbitrarily shaped inclusions, subject to clamped-edge conditions. We construct the first band surfaces of several geometries (including circular cavities) and examine several configurations of elliptical geometries. We are able to demonstrate the existence of ultra flat band surfaces, platonic polarisers, and show that the curvature of the first band surface for compact geometries can be approximately modelled by a circular inclusion occupying the same enclosed area.

5.5 Publication
Flexural wave filtering and platonic polarisers in thin elastic plates

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Abstract: A solution method is given to the problem of plane wave propagation through one- and two-dimensional platonic arrays. The problem is formulated in terms of boundary integral equations and a solution is constructed using boundary element methods. Previous work has been restricted to simple geometries such as circles, pins and squares, and here the framework is extended to consider scatterers of arbitrary shape subject to clamped-edge boundary conditions at the edge. This is done by constructing scattering matrices for a single grating and using Bloch’s theorem to form an eigenvalue problem which connects the grating problem to the array problem. The associated eigenvalues then permit the construction of band surfaces which reveal the flexural wave filtering capabilities of different geometries, as well as the behaviour of Bloch waves within the array. Multiple geometries are investigated and the first band surfaces are computed for these specific cases.

5.5.1 Introduction

Regularly arranged structures which guide and disperse wave energy are important tools with wide applications in science and engineering. These range from the design of spectrometers [113], to the construction of floating offshore wave energy generators [127], the control of sound waves by arrays of trees [92], and are even used to determine the behaviour of ice floes in the marginal ice zone [11]. They are also especially relevant in the study of photonic crystals, phononic crystals, plasmonic crystals (i.e. [55, 57, 163]), and most recently, platonic crystals [6, 41, 106, 107, 124], which are considered here.

Platonic crystals (PlaCs) are defined as periodic structures which guide the propagation of flexural, or bending, waves through thin elastic plates [99]. Here we investigate elastic wave propagation through an infinite stack of platonic gratings which are constructed using scatterers of arbitrary geometry (note that the term platonic refers to any structure embedded in a thin plate, and that a grating is defined as a one-dimensional array). Previously, the bulk of the research into platonic structures has considered only circular scatterers [106, 107] and pins.
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[38, 53, 136] using multipole techniques, or square cavities [41] using finite element models. Here the solution is obtained using boundary integral equations, in an extension to previous work by the authors on single arbitrary scatterers and finite clusters of pins [104, 135, 136]. Other examples of platonic crystals that have received considerable attention include plates that possess substructures, such as arrays of line stiffeners, and arrays of line attachments [48, 83, 84, 101, 102].

In contrast to the Helmholtz equation which governs most time-harmonic wave propagation (e.g. in acoustics and electromagnetic theory), platonic crystals are governed by the fourth-order biharmonic equation. It can be shown straightforwardly that the biharmonic operator decomposes into a product of Helmholtz and modified Helmholtz operators. The presence of this additional modified Helmholtz operator gives rise to an additional evanescent wave field component to the solution, which in certain cases permits strong trapping behaviours inside these periodic structures compared to the Helmholtz equation alone [136]. In fact, the presence of evanescent modes has also been shown to assist with coupling to slow light across waveguides in photonics [142]. It is possible to restrict the solution outline here for the biharmonic equation and consider the associated problem for the Helmholtz equation from inspection, due to the linearity present in both our operator and boundary conditions. However we should note that integral representations for the problem of both a single scatterer and a periodic array, governed by the Helmholtz equation, have also been investigated previously [9, 10].

To demonstrate the behaviour of flexural waves inside the PlaC we will construct band surfaces for a number of different geometries. These band surfaces are a useful visual tool for determining the group velocity inside a PlaC and reveal the incident frequencies and angles where wave propagation through an array is possible. These different geometries can alter the curvature of band surfaces and can therefore filter incident waves in very different ways. For simplicity, we consider only cavities with clamped boundary conditions, which have been shown by Poulton et al. [124] and Movchan et al. [106] to have flat band surfaces and large band gaps, relative to the surfaces associated with free-edge boundary conditions.

For the problem of a single grating one can use Green’s second identity to form a system of boundary integral equations to determine the unknown displacements and normal derivatives at the edge of the central scatterer. These can then be solved using boundary element methods. It is also possible to obtain the solution to the arbitrary shape problem using finite element methods, as demonstrated in Farhat et al. [41] and Dossou et al. [31], however there are circumstances where the boundary element method is the preferred approach, for example, at high wave number values. That said, the direct representation of the quasiperiodic Green’s function is poorly convergent and requires suitable acceleration. A comparison of the various methods used to evaluate the quasiperiodic Green’s function for the Helmholtz equation (away from Wood anomalies - see below) is given in Linton [77], along with tables of values which can be used for numerical validation. The relationship between the quasiperiodic Green’s function for a single grating and a doubly periodic array of scatterers is given in McPhedran et al. [98], which also
The phenomenon of Wood anomalies, or grating resonances, is well known and was first experimentally observed by Wood [160] and explained theoretically by Rayleigh [126]. They correspond to the excitation of a propagating wave which travels along the grating, rapidly decaying in the direction away from the interface, but ultimately propagating to infinity parallel to the interface [13]. They result in the quasiperiodic Green’s function for the grating becoming singular, irrespective of the representation used. There is ongoing research on the effective evaluation of the quasiperiodic Green’s function in the vicinity of these Wood anomalies, including recent work by Barnett and Greengard [9], who use contour deformation, and Bruno and Delourme [20] who propose using modified Green’s functions. For our purposes, when constructing band surfaces we simply remove points that correspond to Wood anomalies from our calculations and use interpolation to reconstruct any missing segments of the band surface.

Note that when using boundary integral methods for the case of a single scatterer, we encounter the problem of irregular frequencies [135]. These irregular frequencies correspond to an adjoint interior problem whose boundary conditions are unknown and are completely non-physical. For the case of a single scatterer we have a clear interior and exterior domain, however for any array of scatterers, irregular frequencies are not present as there is no finite interior domain.

The band surfaces themselves are constructed using the procedure outlined in Botten et al. [16]. That is, once the solution around the edge of the central scatterer is known, we construct the reflection and transmission matrices for a single grating, as discussed in [107]. Once the reflection and transmission matrices for a single grating have been calculated, the extension to the problem of a doubly periodic array reduces to that of a generalised eigenvalue problem as these matrices completely characterise the Bloch waves that can propagate through an array to infinity [16].

For the particular case of circular or pinned scatterers, it is possible to obtain multipole solutions and subsequently the reflection and transmission coefficients can be written as closed-form expressions [107]. From this, one can then follow the scattering matrix approach here to construct the associated band surfaces. Alternatively, the dispersion relation for the plate can be written in terms of lattice sums [106, 136] or trigonometric functions [137] and solved directly to obtain the band surfaces.

A number of geometries are investigated, and we find that large aspect ratios (i.e., long, slender scatterers) are associated with highly anisotropic behaviour as well as preferential directions of propagation along the $x$ and $y$ axis for different wave number intervals. Such structures are referred to here as platonic polarisers and have been identified for the first time here. We are also able to demonstrate the existence of ultraflat bands, which are defined as band surfaces that feature near-zero curvature. Such surfaces give rise to low group velocities and are connected with the concepts of slow and frozen light in photonics [51, 52]. Such ultra low group velocities
can lead to large delays in signal propagation through arrays, which may prove useful in network buffering and other device applications (e.g., in enhanced nonlinear platonic crystals). Other features of these band surfaces include intervals of negative refraction, and wide band gaps above and below the first surface.

The outline of the paper is as follows. In Section 5.5.2 we provide a problem outline and introduce the quasiperiodic Green’s functions for the Helmholtz and modified Helmholtz operators. In Section 5.5.3 we present the boundary element solution for clamped-edge boundary conditions which allows us to retrieve the solution at the edge of the central scatterer. In Section 5.5.4, representations for the quasiperiodic Green’s functions are given in terms of grating sums, and attention is paid to the acceleration of the associated Schlömilch series. In Section 5.5.5 we compute the reflection and transmission coefficients necessary to construct the associated matrices for the case of a single grating. In Section 5.5.6 the conservation of energy relation calculated in Movchan et al. [107] for a platonic grating is shown in matrix form. In Section 5.5.7 we outline the procedure for constructing the band surface of a two-dimensional square array in terms of the reflection and transmission matrices for a single grating. This is followed in Section 5.5.8 by a survey of first band surfaces for a number of different geometries, the energy reflection and transmission by a single grating for a number of geometries, and an investigation into the existence of a perfectly flat band for the case of an elliptical scatter. Appendix 5.5.10 presents the solution for the specific case of a circular scatterer for comparison and additional numerical validation.

5.5.2 Problem formulation

We begin with the biharmonic plate equation in the frequency domain which is given by:

\[(\Delta^2 - k^4)w = (\Delta + k^2)(\Delta - k^2)w = 0,\]  \hspace{1cm} (S.1)

where \(w\) is the plate displacement, \(k^2 = \omega \sqrt{\rho h/D}\), \(\omega\) is the angular frequency, \(\rho\) is the mass density, \(h\) is the thickness and \(D\) is the flexural rigidity of the plate. Here we have assumed a time dependence of \(e^{-i\omega t}\).

As the biharmonic equation is linear and decomposes into a product of Helmholtz and modified Helmholtz operators, we can accordingly decompose the displacement as follows:

\[w = w^H_I + w^H_S + w^M_I + w^M_S.\]  \hspace{1cm} (S.2)
We consider here incident plane waves of the form

\[ w^H_I = \frac{\delta_0}{\sqrt{|\chi_0|}} e^{i\alpha_0 x - i\chi_0 y}, \] (S.3a)

\[ w^M_I = \frac{\hat{\delta}_0}{\sqrt{|\hat{\chi}_0|}} e^{i\alpha_0 x - i\hat{\chi}_0 y}, \] (S.3b)

where \( \delta_0 \) denotes the amplitude of a Helmholtz incident wave, \( \hat{\delta}_0 \) denotes the amplitude of a modified Helmholtz incident wave, \( \alpha_0 = k \sin \theta_i \), \( \chi_0 = k \cos \theta_i \), \( \hat{\chi}_0 = it_0 \), and i.e. \( \chi_0 = it_0 \) where \( t_0 > 0 \). Note that the incident potentials above have inverse square root amplitude prefactors. These are introduced so that our conservation of energy relation (S.36) is appropriately normalised. Furthermore, we consider only one incident wave type at a given time (e.g., \( \delta_0 = 1 \) and \( \hat{\delta}_0 = 0 \)).

We now construct the boundary equations for a platonic grating comprised of identically spaced scatterers of arbitrary geometry. If the scatterers are circular, the solution can be found using multipole techniques (see Appendix 5.5.10). Using Green’s second identity in a manner similar to Smith et al. [135], the plate displacement can be expressed by the following system of boundary integral equations

\[
\frac{1}{2} w^M(x) = w^M_I(x) + \sum_{m=-\infty}^{\infty} \int_{\partial \Omega_m} \left\{ \partial_{n'} G^M(x, x') w^M(x') - \partial_{n'} w^M(x') G^M(x, x') \right\} dS', \quad \text{(S.4a)}
\]

\[
\frac{1}{2} w^H(x) = w^H_I(x) + \sum_{m=-\infty}^{\infty} \int_{\partial \Omega_m} \left\{ \partial_{n'} G^H(x, x') w^H(x') - \partial_{n'} w^H(x') G^H(x, x') \right\} dS', \quad \text{(S.4b)}
\]

for all \( x \in \partial \Omega_m \) where \( m \in \mathbb{Z} \). Here \( G^H(x, x') = \frac{i}{2} H^{(1)}_0(kr) \) and \( G^M(x, x') = \frac{i}{2} H^{(1)}_0(ikr) \) denote the Green’s functions for the Helmholtz and modified Helmholtz equations respectively, which are expressed in terms of zero-order Hankel functions, with \( r = |x - x'| \) denoting the distance between the field and source points respectively. The normal derivatives are defined as \( \partial_{n'} = (\partial_{x'}, \partial_{y'}) \cdot \mathbf{n}' \) where \( \mathbf{n}' \) is the normal unit vector to the boundary, pointing outwards. We denote the boundary of the \( m \)th smooth cavity in our infinite array by \( \partial \Omega_m \).

As the grating is regularly structured, the variation in the displacement between adjacent bodies can be represented by the quasiperiodicity condition:

\[
w(x' + (md, 0)) = w(x'_m) = e^{i\alpha_0 md} w(x'), \quad \text{(S.5)}
\]

where \( d \) denotes the period of the grating. Additionally the field point can be restricted to the central scatterer, and consequently (S.4a) and (S.4b) can be represented by the more compact
system

\[
\frac{1}{2} w^M(x) = w^M_1(x) + \int_{\partial \Omega_0} \left\{ \partial_{n'} G^M_g(x, x') w^M(x') - \partial_{n'} w^M(x') G^M_g(x, x') \right\} dS', \quad (S.6a)
\]

\[
\frac{1}{2} w^H(x) = w^H_1(x) + \int_{\partial \Omega_0} \left\{ \partial_{n'} G^H_g(x, x') w^H(x') - \partial_{n'} w^H(x') G^H_g(x, x') \right\} dS', \quad (S.6b)
\]

for \( x \in \partial \Omega_0 \), where we have the quasiperiodic Green’s functions

\[
G^H_g(x, x') = \sum_{m=-\infty}^{\infty} G^H_g(x, x'_m) e^{ia_0 md}, \quad (S.7a)
\]

\[
\partial_{n'} G^H_g(x, x') = \sum_{m=-\infty}^{\infty} \partial_{n'} G^H_g(x, x'_m) e^{ia_0 md}, \quad (S.7b)
\]

\[
G^M_g(x, x') = \sum_{m=-\infty}^{\infty} G^M_g(x, x'_m) e^{ia_0 md}, \quad (S.7c)
\]

\[
\partial_{n'} G^M_g(x, x') = \sum_{m=-\infty}^{\infty} \partial_{n'} G^M_g(x, x'_m) e^{ia_0 md}. \quad (S.7d)
\]

Here the subscript \( g \) is introduced to represent a Green’s function for a grating as opposed to a conventional Green’s function. We consider the simplest possible condition that can be imposed on a thin plate – when the edges of the perforations are clamped.

### 5.5.3 Clamped-edge boundary conditions

For thin plates the boundary conditions for a scatterer with clamped-edge conditions are given by

\[
w(x') = 0, \quad \text{and} \quad \partial_{n'} w(x') = 0, \quad \text{for} \quad x' \in \partial \Omega_0. \quad (S.8)
\]

This implies that

\[
w^H = -w^M, \quad (S.9a)
\]

\[
\partial_{n'} w^H = -\partial_{n'} w^M, \quad (S.9b)
\]

which provides the necessary coupling condition to solve the system (S.6a) and (S.6b) via some numerical scheme, such as boundary element methods (BEM). Such methods begin by subdividing the boundary of the scatterer into \( q \) panels (with each panel denoted by \( \partial \Omega_{0q} \)) of approximately constant length over which the associated displacement and normal derivative is assumed constant. This admits the block-matrix system:

\[
\begin{bmatrix}
\frac{1}{2} I & A^M \\
\frac{1}{2} I & A^H
\end{bmatrix}
\begin{bmatrix}
B^M \\
B^H
\end{bmatrix}
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
= f \quad (S.10)
\]
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where \( \xi_p = w^H(x_p), \eta_p = \partial_{n'} w^H(x_p), \)

\[
A_{pq}^{HM} = \int_{\partial \Omega_0} \partial_{n'} G_g^M H(x_p, x') dS',
\]

\[
B_{pq}^{HM} = \int_{\partial \Omega_0} G_g^M H(x_p, x') dS',
\]

(S.11a)

(S.11b)

and \( \mathbf{f} = [0, \xi^i]^T \) or \( \mathbf{f} = [\eta^i, 0]^T \) depending on the choice of the incident wave potential. Here \( \xi_p^i = w^H(x_p), \eta_p^i = -w^M(x_p), I \) denotes the identity matrix and \( x_p \) denotes the midpoint of the \( p \)th panel.

Note that the quasiperiodic Green’s functions (S.7) above are singular at \( r \equiv 0 \). Numerically we overcome this by adding and subtracting the log singularity that is encountered by the zero-order Hankel functions as \( kr \to 0 \) in a procedure known as the singularity subtraction method [161]. In contrast, the normal derivatives of these Green’s functions are smooth across \( r \equiv 0 \) and so interpolation is used to recover values at \( r \equiv 0 \) when constructing the \( A \) and \( B \) matrices.

Furthermore, the Green’s function for the Helmholtz equation is well known to have slow convergence and needs to be accelerated by an efficient numerical scheme. This is the topic of the following section.

5.5.4 Acceleration of the Green functions

The technique outlined below is from Nicorovici and McPhedran [109] and is included here for completeness. We begin by considering the function:

\[
G_g^H(x, x') = \sum_{m=-\infty}^{\infty} G_g^H(x, x'_m)e^{i\lambda_md}
\]

(S.12)

\[
= \frac{i}{4} \sum_{m=-\infty}^{\infty} H_0^{(1)} \left( k \sqrt{(x-x'-md)^2 + (y-y')^2} \right) e^{i\lambda_md}.
\]

The expression (S.12) can be rewritten using Graf’s addition theorem

\[
C_{\nu}(w)e^{i\nu\chi} = \sum_{l=-\infty}^{\infty} C_{l+\nu}(u)J_l(v)e^{il\alpha}, \quad \text{where} \quad |ve^{i\alpha}| < |u|,
\]

(S.13)

(formula (9.1.79) in Abramowitz and Stegun [1]) within the fundamental cell \( (x, y) = [-d/2, d/2] \times [-d/2, d/2] \), where \( C_{\nu} \) denotes an arbitrary Bessel function. Here the lines \( u, v \) and \( w \) form a closed triangle with \( w = \sqrt{u^2 + v^2 - 2uv \cos \alpha}, \alpha \) denoting the angle between the lines \( u \) and \( v \), and \( \chi \) denoting the angle between the lines \( u \) and \( w \). This admits the representation

\[
G_g^H(x, x') = \frac{i}{4} \left[ H_0^{(1)}(kr) + S_0^{HG} J_0(kr) + 2 \sum_{l=1}^{\infty} S_l^{HG} J_l(kr) \cos(l\theta) \right],
\]

(S.14)
where we introduce the grating sum
\[ S_t^{H,G}(\alpha_0, k, d) = \sum_{n \neq 0} H_{2}(\pi|kd|e^{i\alpha_0 nd}e^{ik\phi_n}). \] (S.15a)

Here \( \phi_n = \pi H(-n) \), \( H(x) \) denotes a Heaviside function, and \((r, \theta)\) are polar coordinates with \( r = \sqrt{(x - x')^2 + (y - y')^2} \) and \( \theta = \tan^{-1}((y - y')/(x - x')). \) The grating sum \( S_t^{H,G} \) in (S.15a) has a radius of convergence \( r = d \), and can be decomposed straightforwardly into even and odd index grating sums:

\[ S_{2l}^{H,G}(\alpha_0, k, d) = 2 \sum_{n=1}^{\infty} H_{2l}^{(1)}(nk) \cos(\alpha_0 nd), \] (S.16a)

\[ S_{2l-1}^{H,G}(\alpha_0, k, d) = 2i \sum_{n=1}^{\infty} H_{2l-1}^{(1)}(nk) \sin(\alpha_0 nd). \] (S.16b)

Series of this type are known as Schlomilch series and are notorious for poor convergence [153]. Directly convergent expressions for these Schlomilch series can be found in Twersky [153], however we include below (2.49), (2.53) and (2.54) from Linton [77] which are the result of applying Kummer’s method to the expressions in [153]:

\[ S_0^{H,G} = -1 - \frac{2i}{\pi} \left[ \gamma + \log \left( \frac{kd}{4\pi} \right) \right] + \frac{2}{\chi_0 d} - \frac{id^2(k^2 + 2\alpha_0^2)}{4\pi^3} \zeta(3) \]
\[ - \frac{2i}{d} \sum_{m \in \mathbb{Z}\setminus\{0\}} \left( \frac{i}{\chi_m} - \frac{d}{2\pi|m|} - \frac{d^3(k^2 + 2\alpha_0^2)}{16\pi^3|m|^3} \right), \] (S.17a)

\[ S_{2l}^{H,G} = \frac{2e^{-2i\theta_0}}{\chi_0 d} + 2 \sum_{m=1}^{\infty} \left\{ \frac{e^{-2i\theta_m}}{\chi_m d} + \frac{e^{2i\theta_m}}{\chi_m d} + \frac{i(-1)^l}{m\pi} \left( \frac{kd}{4m\pi} \right)^{2l} \right\} \]
\[ - \frac{2i(-1)^l}{\pi} \left( \frac{kd}{4\pi} \right)^{2l} \zeta(2l + 1) + \frac{i}{l\pi} \]
\[ + \frac{i}{\pi} \sum_{m=1}^{l} \frac{(-1)^m 2^m (l + m - 1)!}{(2m)! (l - m)!} \left( \frac{2\pi}{kd} \right)^{2m} B_{2m} \left( \frac{\alpha_0 d}{2\pi} \right), \] (S.17b)

\[ S_{2l-1}^{H,G} = \frac{-2e^{-i(2l-1)\theta_0}}{\chi_0 d} - 2 \sum_{m=1}^{\infty} \left\{ \frac{e^{-i(2l-1)\theta_m}}{\chi_m d} + \frac{e^{i(2l-1)\theta_m}}{\chi_m d} + \frac{\alpha_0 d(-1)^l}{m^2\pi^2} \left( \frac{kd}{4m\pi} \right)^{2l-1} \right\} \]
\[ - \frac{2\alpha_0 d(-1)^l}{\pi^2} \left( \frac{kd}{4\pi} \right)^{2l-1} \zeta(2l + 1) \]
\[ - \frac{2}{\pi} \sum_{m=0}^{l-1} \frac{(-1)^m 2^m (l + m - 1)!}{(2m+1)! (l - m - 1)!} \left( \frac{2\pi}{kd} \right)^{2m+1} B_{2m+1} \left( \frac{\alpha_0 d}{2\pi} \right), \] (S.17c)
where \( \zeta(z) \) represents the Riemann zeta function, \( \gamma \) denotes the Euler–Mascheroni constant, \( B_n(z) \) represents a Bernoulli polynomial of order \( n \),

\[
\alpha_n = k \sin(\theta_n) = \alpha_0 + \frac{2\pi n}{d}, \quad \text{and} \quad \chi_n = \begin{cases} 
\sqrt{k^2 - \alpha_n^2}, & \alpha_n^2 \leq k^2 \\
-i\sqrt{\alpha_n^2 - k^2}, & \alpha_n^2 > k^2.
\end{cases}
\] (S.18)

The derivatives of the Green’s function above are given by

\[
\partial_{\xi'}G_H^H = -\frac{ik}{4} \left[ H_1^{(1)}(kr)\partial_{\xi'} r + S_0^{H,G} J_1(kr)\partial_{\xi'} r 
+ \sum_{l=1}^{\infty} S_l^{H,G} \left\{ (I_{l+1}(kr) - I_{l-1}(kr)) \partial_{\xi'} r \cos(l\theta) + \frac{2l}{k} J_1(kr) \sin(l\theta) \partial_{\xi'} \theta \right\} \right].
\] (S.19)

where \( \xi' = x' \) or \( \xi' = y' \) as required to compute \( \partial_{\xi'} G_H^H \).

In general, the Green’s function representations (S.7c) and (S.7d) for the modified Helmholtz equation \( G^M_g \) can be computed straightforwardly as the summand decays exponentially for \( |m| \to \infty \). However for consistency one can express the quasiperiodic Green’s function for the modified Helmholtz case as:

\[
G^M_g(x, x') = \frac{1}{2\pi} \left[ K_0(kr) + S_0^{K,G} I_0(kr) + 2 \sum_{l=1}^{\infty} S_l^{K,G} I_l(kr) \cos(l\theta) \right],
\] (S.20)

where

\[
S_l^{K,G} = \sum_{n \neq 0} K_l(\nkd) e^{i\alpha_0 nd} e^{il\phi_n},
\] (S.21)

or after decomposition into even and odd orders,

\[
S_{2l}^{K,G} = 2 \sum_{n=1}^{\infty} K_{2l}(nkd) \cos(\alpha_0 nd),
\] (S.22a)
\[
S_{2l-1}^{K,G} = 2i \sum_{n=1}^{\infty} K_{2l-1}(nkd) \sin(\alpha_0 nd).
\] (S.22b)

The grating sums for the modified Helmholtz equation are in general, rapidly convergent and directly evaluable [107]. The derivatives for the modified Green’s function can be obtained via

\[
\partial_{\xi'}G^M_g = \frac{k}{2\pi} \left[ -K_1^{(1)}(kr)\partial_{\xi'} r + S_0^{K,G} I_1(kr)\partial_{\xi'} r 
+ \sum_{l=1}^{\infty} S_l^{K,G} \left\{ (I_{l+1}(kr) + I_{l-1}(kr)) \partial_{\xi'} r \cos(l\theta) - \frac{2l}{k} I_1(kr) \sin(l\theta) \partial_{\xi'} \theta \right\} \right],
\] (S.23)

where \( \xi' \) is defined as above.

Care must be taken with the quasiperiodic Green’s function for the Helmholtz equation as it
fails to converge at particular combinations of $k$ and $\alpha_0$ which correspond to Wood anomalies [126, 160]. Wood anomalies occur when a diffraction order switches from being evanescent to propagating, i.e. when $\chi_m \equiv 0$ for some $m \in \mathbb{Z}$. It is also possible to have double Wood anomalies which correspond to $\alpha_m = -\alpha_n = k$ for some $m, n \in \mathbb{Z}$ and occur when $kd/2\pi \in \mathbb{Z}$ [9].

5.5.5 Reflection and Transmission Coefficients

We follow a similar procedure to the one outlined in Movchan et al. [107], except here we generalise for bodies of arbitrary shape. However it is first necessary to introduce appropriate notation in order to construct the reflection and transmission matrices for our platonic crystal.

In particular, we introduce the notation $w^{HH}$ and $w^{MH}$ to denote the $w^H$ and $w^M$ components of the displacement (respectively), when a Helmholtz incident wave $w^H_I$ is present. We also introduce $w^{MM}$ and $w^{HM}$ to denote $w^M$ and $w^H$ (respectively), for the modified Helmholtz incident wave $w^M_I$. That is, the second superscript refers to the incident wave type and the first refers to the displacement component. It is necessary to consider all these possible combinations as modified Helmholtz incident waves give rise to Helmholtz waves and vice versa.

We begin by applying Green’s second identity around the central cell as shown in Figure 5.14, using two special test functions along with the displacements $w^{HH}$, $w^{HM}$, $w^{MH}$ and $w^{MM}$. The test functions are defined as

$$v^H_m = \frac{1}{\sqrt{|\mu_m|}} e^{-i\alpha_m x + i\mu_m y},$$

$$v^M_m = \frac{1}{\sqrt{|\mu_m|}} e^{-i\alpha_m x + i\hat{\mu}_m y},$$

where $\mu_m = \pm \chi_m$ and $\hat{\mu}_m = \pm \hat{\chi}_m$, which are specified later.

When a Helmholtz incident wave is considered the total wave field on the top of the fundamental cell ($x \in \gamma_+$) can be expressed in plane wave form as

$$w = w^H_I + \sum_p \frac{R^H_p}{|\chi_p|} e^{i\alpha_p x + i\chi_p y} + \sum_p \frac{\hat{R}^H_p}{|\chi_p|} e^{i\alpha_p x + i\hat{\chi}_p y},$$

where $R^H_p$ and $\hat{R}^H_p$ denote the reflection coefficients for the Helmholtz and modified Helmholtz fields respectively. This can be decomposed in a similar manner into $w = w^{HH} + w^{MH}$ where

$$w^{HH}(x) = w^H_I + \sum_p \frac{R^H_p}{|\chi_p|} e^{i\alpha_p x + i\chi_p y}, \quad w^{MH}(x) = \sum_p \frac{\hat{R}^H_p}{|\chi_p|} e^{i\alpha_p x + i\hat{\chi}_p y}.$$
Likewise for a modified incident wave the total field at the top of the cell is given by
\[
w = w_1^M + \sum_p \frac{R_p^M}{\sqrt{|\chi_p|}} e^{i\alpha_p x + i\chi_p y} + \sum_p \frac{T_p^H}{\sqrt{|\chi_p|}} e^{i\alpha_p x - i\hat{\chi}_p y},
\] (S.26a)

where \(R_p^M\) and \(\hat{T}_p^H\) denote the reflection coefficients for the Helmholtz and modified Helmholtz fields respectively. This can be decomposed into
\[
w = w_{HM}(x) + w_{MM}(x),
\] (S.27)

and for a modified Helmholtz wave we have
\[
w = \sum_p \frac{T_p^H}{\sqrt{|\chi_p|}} e^{i\alpha_p x - i\hat{\chi}_p y} + \sum_p \frac{T_p^M}{\sqrt{|\chi_p|}} e^{i\alpha_p x - i\hat{\chi}_p y} = w_{HH}(x) + w_{MM}(x),
\] (S.28)

where \(T_p^H, T_p^M\) correspond to the Helmholtz component and \(\hat{T}_p^H, \hat{T}_p^M\) correspond to the modified Helmholtz component of the field.

Applying Green’s second identity to the central cell with our first test function (S.24a) and first pairing (H-H) reveals
\[
\int_{\partial \Omega_0, \gamma_+ \gamma_-} \left\{ v_m \partial_n w_{HH} - w_{HH} \partial_n v_m^H \right\} dS' = 0.
\] (S.29a)
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Note that these test functions ((S.24a) and (S.24b)) have the opposite quasiperiodicity to $w^{HH}$, $w^{HM}$, $w^{MH}$ and $w^{MM}$ and thus there is no contribution from $\gamma_l$ and $\gamma_r$ (the vertical sides of the cell).

It can be shown straightforwardly that the integral along $\gamma_+$ has the contribution

$$
\int_{\gamma_+} \left\{ v_m^H(x') \partial_{n'} w^{HH}(x') - w^{HH}(x') \partial_{n'} v_m^H(x') \right\} \, ds' =
$$

$$
d \left[ \frac{-i(\chi_0 + \mu_0)}{\sqrt{|\chi_0| \sqrt{|\mu_0|}}} e^{-i(\chi_0 - \mu_0)y'} \delta_{m0} + \frac{iR_m^H(\chi_m - \mu_m)}{\sqrt{|\chi_m| \sqrt{|\mu_m|}}} e^{i(\chi_m + \mu_m)y'} \right],
$$

(S.29b)

where $y' = d/2$, and similarly that $\gamma_-$ has the contribution

$$
\int_{\gamma_-} \left\{ v_m^H(x') \partial_{n'} w^{HH}(x') - w^{HH}(x') \partial_{n'} v_m^H(x') \right\} \, ds' =
$$

$$
d \left[ \frac{-i(\chi_0 + \mu_0)}{\sqrt{|\chi_0| \sqrt{|\mu_0|}}} e^{-i(\chi_0 - \mu_0)y'} \delta_{m0} + \frac{iR_m^H(\chi_m - \mu_m)}{\sqrt{|\chi_m| \sqrt{|\mu_m|}}} e^{i(\chi_m + \mu_m)y'} \right],
$$

(S.29c)

where $y'' = -d/2$. Therefore (S.29a) is of the form:

$$
\int_{\partial\Omega_0} \left( v_m^H(x'; \chi_m) \partial_{n'} w^{HH}(x') - w^{HH}(x') \partial_{n'} v_m^H(x'; \chi_m) \right) \, ds' = 0,
$$

(S.29d)

where the final integral is oriented anticlockwise, and must be evaluated numerically as it depends on the geometry of the scatterer. Specifying $\mu_m = \chi_m$ for all $m$ and $\mu_m = -\chi_m$ for all $m$ reveals

$$
T_m^H = \delta_{m0} + \frac{|\chi_m|}{2i d \chi_m} \int_{\partial\Omega_0} \left( v_m^H(x'; \chi_m) \partial_{n'} w^{HH}(x') - w^{HH}(x') \partial_{n'} v_m^H(x'; \chi_m) \right) \, ds',
$$

(S.30a)

$$
R_m^H = \frac{|\chi_m|}{2i d \chi_m} \int_{\partial\Omega_0} \left( v_m^H(x'; -\chi_m) \partial_{n'} w^{HH}(x') - w^{HH}(x') \partial_{n'} v_m^H(x'; -\chi_m) \right) \, ds',
$$

(S.30b)

respectively. Similarly, for the M-M case we use the second test function (S.24b) and specify $\mu_m = \widehat{\chi}_m$ and $\mu_m = -\widehat{\chi}_m$ to obtain

$$
\widehat{T}_m^M = \delta_{m0} - \frac{1}{2d} \int_{\partial\Omega_0} \left( v_m^M(x'; \widehat{\chi}_m) \partial_{n'} w^{MM}(x') - w^{MM}(x') \partial_{n'} v_m^M(x'; \widehat{\chi}_m) \right) \, ds',
$$

(S.31a)

$$
\widehat{R}_m^M = \frac{1}{2d} \int_{\partial\Omega_0} \left( v_m^M(x'; -\widehat{\chi}_m) \partial_{n'} w^{MM}(x') - w^{MM}(x') \partial_{n'} v_m^M(x'; -\widehat{\chi}_m) \right) \, ds',
$$

(S.31b)

respectively. Integrating around the fundamental cell for the H-M case with the first test function
Using the expressions above, one can construct $R^{(0)}$ and $T^{(0)}$ which denote the basic reflection and transmission matrices of a single grating for a wave travelling in the direction $y < 0$ (i.e., from above). These matrices are given by

$$ R^{(0)} = \begin{bmatrix} R_{HH}^{(0)} & R_{HM}^{(0)} \\ R_{MH}^{(0)} & R_{MM}^{(0)} \end{bmatrix}, \quad \text{and} \quad T^{(0)} = \begin{bmatrix} T_{HH}^{(0)} & T_{HM}^{(0)} \\ T_{MH}^{(0)} & T_{MM}^{(0)} \end{bmatrix}, $$

(S.34)

and have a phase origin at the centre of each scatterer (note the phase origin denotes the point from which phases are measured). Here the $(p, q)^{th}$ element of $R_{HH}^{(0)}$ is $R_{p}^{H}$ which corresponds to the incident wave $w_1^{H}$ travelling down the channel, or grating order, $\alpha_q = \alpha_0 + 2\pi q/d$. Likewise, the $(p, q)^{th}$ element of $R_{MH}^{(0)}$ is $R_{p}^{H}$ corresponding to $w_1^{H}$ down the channel $\alpha_q$. Finally, the $(p, q)^{th}$ elements of $R_{HM}^{(0)}$ and $R_{MM}^{(0)}$ are $R_{p}^{M}$ and $R_{p}^{M}$ respectively, corresponding to $w_1^{M}$ down channel $\alpha_q$. The transmission matrix $T^{(0)}$ is created in an analogous manner to the definitions above.

Once formed, these matrices can then be shifted to the appropriate phase origins at $x \in \gamma^+$ and $x \in \gamma^-$ via the appropriate matrix multiplication:

$$ T = PT^{(0)}P, \quad \text{and} \quad R = PR^{(0)}P, $$

(S.35a)

where

$$ P = \begin{bmatrix} P^H \\ 0 \end{bmatrix}, \quad P_{pq}^H = \delta_{pq} e^{i\gamma_p d/2}, \quad \text{and} \quad P_{pq}^M = \delta_{pq} e^{i\pi_p d/2}. $$

(S.35b)

5.5.6 Energy balance equations

As a check on numerics we use the previously computed conservation of energy relation as given in equation (7.3) of Movchan et al. [107] for a single grating. We express it in the following matrix form:

$$ R^T U R^* + T^T U T^* = U - i \left( V R^* - R^T V^T \right), $$

(S.36)
where

\[
U = \begin{bmatrix} U^H & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} V^H & 0 \\ 0 & V^M \end{bmatrix},
\]

and

\[
U^H_{pq} = \beta_p \delta_{pq}, \quad V^H_{pq} = (\beta_p - 1) \delta_{pq}, \quad V^M_{pq} = \delta_{pq}, \quad \beta_n = \begin{cases} 1, & \alpha_n^2 \leq k^2 \\ 0, & \alpha_n^2 > k^2 \end{cases},
\]

and star notation represents a complex conjugate operation. The relation (S.36) is satisfied independent of the order of truncation, as our method of interface matching in Section 5.5.5 ensures the fluxes are matched perfectly [14, 15].

5.5.7 Platonic band surface calculation

The procedure for calculating band surfaces using scattering matrices is outlined in Botten et al. [16] for the case of a general photonic crystal array, and a brief outline is included here for the specific case of a square lattice.

We begin by constructing horizontal lines above and below the central grating which sit halfway between the subsequent layers, and coincide with the top and bottom of the fundamental cell discussed earlier. These lines are introduced in order to shift the phase origin from the centre of each scatterer to \((x, y) = (x, \pm d/2)\). We introduce \(p_1\) and \(p_2\) to represent these two interfaces (above and below, respectively) and note that there are fields travelling across each line in the directions \(y > 0\) and \(y < 0\) (denoted by \(\pm\) respectively). This admits the following representation of the displacement at each interface which is given by:

\[
w^j(x) = \sum_{p=\pm \infty}^{\infty} \frac{1}{\sqrt{|\chi_p|}} \left( f_p^{(j)} e^{-i \chi_p (y-y_j)} + f_p^{(j)+} e^{i \chi_p (y-y_j)} \right) e^{i \alpha_p x} + \frac{1}{\sqrt{|\hat{\chi}_p|}} \left( g_p^{(j)} e^{-i \hat{\chi}_p (y-y_j)} + g_p^{(j)+} e^{i \hat{\chi}_p (y-y_j)} \right) e^{i \alpha_p x} \quad (S.38)
\]

where \(j = 1\) and \(j = 2\) denote the lines \(p_1\) and \(p_2\) respectively and \(f_p^{(j)\pm}, g_p^{(j)\pm}\) denote the wave amplitudes of the Helmholtz and modified Helmholtz wave components respectively.

Introducing the column vector \(h^j = [f_p^{(j)+}, g_p^{(j)+}]^T\) for compactness it is then possible to construct a scattering matrix which connects incoming with outgoing fields, i.e.

\[
\begin{bmatrix} h_2^j \\ h_1^j \end{bmatrix} = \begin{bmatrix} T & R' \\ R & T' \end{bmatrix} \begin{bmatrix} h_1^- \\ h_2^- \end{bmatrix}, \quad (S.39)
\]

where \(R\) and \(T\) are the reflection and transmission matrices for an incident wave from above the grating, and \(R'\) and \(T'\) the appropriate matrices for an incident wave from below the grating. We observe that for \(y\)-symmetric scatterers in a square lattice, the matrices corresponding to an incident wave from below are the same as those from above, and so \(R = R'\) and \(T = T'\).
It is also possible to connect the fields travelling up and down the $y$-axis by imposing a quasiperiodicity condition in the $y$-direction:

$$h_2^- = \mu h_1^-,$$

and 

$$h_2^+ = \mu h_1^+,$$  \hspace{1cm} (S.40)

which arises from the fact that there is only a phase difference in the solution across the cell. Here $\mu = e^{-i\kappa_y d}$ and $\kappa_y$ is the unknown $y$-component of the Bloch vector $\mathbf{\kappa} = (\kappa_x, \kappa_y) = (\alpha_0, \kappa_y)$, and consequently (S.39) can be expressed as

$$\begin{bmatrix}
T - \mu I & R \\
R & T - \mu^{-1} I
\end{bmatrix}
\begin{bmatrix}
h_1^- \\
h_2^+
\end{bmatrix} = 0. $$  \hspace{1cm} (S.41)

We then define $P(\mu)$ as the determinant of the above block matrix system and seek values of $\mu$ such that $P(\mu) = 0$. Note that due to the lattice symmetry we have pairings of zeros $P(\mu) = P(1/\mu)$.

Botten et al. [16] then propose an efficient representation of such symmetric geometries by applying a unitary transformation to (S.41), which halves the dimension of the block matrix system (S.41) by converting the system into a pair of equivalent eigenvalue equations. Either of these systems can then be solved to yield the unknown $\mu$, and we specify the first system here for compactness:

$$S^{-1}Tg = \frac{1}{2c} g,$$  \hspace{1cm} (S.42)

where for our purposes $g$ denotes an arbitrary eigenvector, $2c = \mu + \mu^{-1}$, and

$$S = I + (T - R)(T + R).$$  \hspace{1cm} (S.43)

Providing $c \in \mathbb{R}$, then eigenvalues with $|c| \leq 1$ correspond to real or propagating states, and hence, admissible phase factors $\mu$. If $c$ is complex-valued or $|c| > 1$ then we have a phase factor which corresponds to an evanescent or non-propagating state. In other words, once the eigenvalues $\lambda = 1/2c$ have been found, we merely need to solve the simple quadratic equation

$$\mu^2 - 2c\mu + 1 = 0$$

to reveal values of $\mu$ which are associated with propagating Bloch modes.

Solving our eigenvalue problem above reveals eigenvalues $\mu = e^{-i\kappa_y d}$, and so we can easily see that

$$\kappa_y = \text{Re} \left[ \frac{i}{d} \log(\mu) \right],$$

or equivalently, 

$$\kappa_y = -\text{arg}(\mu)/d,$$  \hspace{1cm} (S.44)

which are our unknown Bloch vector elements.

For a scatterer of arbitrary shape (i.e. no $y$-symmetry) the system to be solved is

$$\begin{bmatrix}
T - R'T'^{-1}R & R'T'^{-1} \\
-T'^{-1}R & T'^{-1}
\end{bmatrix}
\begin{bmatrix}
h_1^- \\
h_2^+
\end{bmatrix} = \mu
\begin{bmatrix}
h_1^- \\
h_2^+
\end{bmatrix},$$  \hspace{1cm} (S.45)

which directly reveals the unknown Bloch factors as eigenvalues. That said, computing the
matrix $T^{-1}$ requires some care, as it is less numerically stable than (S.42).

### 5.5.8 Results and discussion

We begin by computing the first band surfaces for a range of scattering geometries which exhibit four-fold, three-fold, and two-fold symmetry, as seen in Figure 5.15. We assume a grating period of $d = 1$, discretise the scatterer boundaries with 700 panels at the edge and consider 2 propagating orders (i.e., $m = -2$ to 2) when constructing the reflection and transmission matrices. We find that this number of panels is sufficient to ensure good convergence of the reflection and transmission coefficients (S.30a) to (S.33b) across all geometries.

All of the scatterers in Figure 5.15 have been created to approximately preserve the area associated with a circle of radius $a = 0.2$. The scale of these geometries is determined using the Shoelace algorithm [167] for a large number of points at the boundary edge and so we introduce the parametric form

$$ (x, y) = (a r(\theta) \cos(\theta), b r(\theta) \sin(\theta)), \quad (S.46) $$

to construct our scatterers. For the circular scatterer in Figure 5.15(a) we simply specify $a = b = 0.2$ and $r(\theta) = 1$, for the L4 geometry shown in Figure 5.15(b) we have

$$ r(\theta) = \left(\frac{1}{\cos^4(\theta) + \sin^4(\theta)}\right)^{1/4}, \quad \text{and} \quad a = b = 0.1841, \quad (S.47a) $$

for the four leaf clover in Figure 5.15(c) we have

$$ r(\theta) = 4 + \cos(4\theta), \quad \text{and} \quad a = b = 0.0493, \quad (S.47b) $$

for the three leaf clover in Figure 5.15(d) we have

$$ r(\theta) = 4 + \cos(3\theta), \quad \text{and} \quad a = b = 0.0493, \quad (S.47c) $$

and for the ellipse ($b/a = 2$) in Figure 5.15(e) we have $a = 0.1414$, $b = 0.2828$ and $r(\theta) = 1$. The final diamond geometry is formed via

$$ (x, y) = \left(\frac{a}{\sqrt{2}} r(\theta) \{\cos(\theta) - \sin(\theta)\}, \frac{3a}{2\sqrt{2}} r(\theta) \{\cos(\theta) + \sin(\theta)\}\right), \quad (S.47d) $$

where $a = 0.1504$ and the $r(\theta)$ expression from (S.47a) is used. Here the rounded square scatterer is referred to as an L4 geometry due to its association with the L4 norm ($x^4 + y^4 = 1$).

In Figure 5.16 we construct the first band surfaces for these geometries. In Figures 5.16(a), 5.16(b) and 5.16(c) we see band surfaces which exhibit qualitatively similar curvatures, which one would expect for geometries that exhibit four-fold symmetry and have aspect ratio close to
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Figure 5.15: Scattering geometries considered: (a) circle, (b) L4 or rounded square, (c) four leaf clover, (d) three leaf clover, (e) ellipse \((b/a = 2)\), and (f) diamond.

1 (i.e., compact geometry). They each feature a maximum at the origin, a total of four minima at \((\kappa_x, \kappa_y) = (\pm \pi, \pm \pi)\) and four saddle points at \((\kappa_x, \kappa_y) = (0, \pm \pi)\) and \((\kappa_x, \kappa_y) = (\pm \pi, 0)\). For the circular scatterer we have a relative maximum value of \(k = 5.6554\) and relative minima values of \(k = 5.5677\), for the L4 geometry we have a maximum value of \(k = 5.7482\) and minima values of \(k = 5.7051\) and for the four-leaf clover a maximum value of \(k = 5.7703\) with minima values of \(k = 5.5549\). The isofrequency contours of these band surfaces are largely circular in nature, and as such the array is said to act as an isotropic medium (that is, the group velocity vector is parallel to the the crystal wave vector [144]). The frequency ranges for these three geometries does vary, and the band surface with the smallest \(k\)-range is the L4 geometry which corresponds to low magnitude group velocity vectors and thus, slower wave energy propagation through the array. The contours for the four leaf clover scatterer in Figure 5.16(c) do exhibit sharper contours near the saddle points, but still remain qualitatively similar to the other shapes exhibiting four-fold symmetry. In Figure 5.16(d) we have the band surface for a three leaf clover shape, which is related to the previous figures and possesses the same number of critical points (with a maximum of \(k = 5.8332\) and minima at \(k = 5.7224\)). However it does not exhibit perfect isotropy as the contours (especially near the origin) are slightly elliptical. Additionally the band surface does not exhibit up-down or left-right symmetry as a result of the scattering geometry and so the contours for \(\kappa_y > 0\) can be obtained by rotation and reflection of the band surface picture obtained for \(\kappa_y < 0\). These first four scatterers all possess regions where negative refraction through their respective arrays is supported, which are characterised by diagonally sloping contours, as seen in [136].

In Figure 5.16(e) we see the band surface for an ellipse \((b/a = 2)\) which exhibits very different
behaviour compared to the previous geometries. This band surface now features two maxima at \((\kappa_x, \kappa_y) = (0, \pm \pi)\), two minima at \((\kappa_x, \kappa_y) = (\pm \pi, 0)\) and saddle points at both the origin and at the extreme corners of the Brillouin zone. The array exhibits extremely anisotropic behaviour, however isotropic contours are seen in the vicinity of the minima. It also exhibits preferential directions of propagation – that is, at low \(k\) values the array (below \(k = 5.6175\)) favours horizontal propagation through the platonic crystal, and at high \(k\) values (above \(k = 5.6175\)), it favours vertical propagation through the crystal. We refer to structures which exhibit strong preferential directions as platonic polarisers, in an analogous manner to their optical counterparts. A similar
picture is seen for the case of a diamond geometry in Figure 5.16(f), except that the contours strongly favour horizontal propagation through the array as opposed to vertical (which is only seen in a small window around $\kappa_x = 0$ for $k > 5.6135$). As with the elliptical scatterer we have two maxima ($k = 5.6609$) and two minima ($k = 5.5241$), as well as saddle points at the origin and extreme corners of the Brillouin zone. If we were to rotate the diamond by 90 degrees to a horizontal position, the associated band surface would also rotate by the same angle. In such an instance, we would see wide intervals where ultrarefraction is supported due to the very flat contours seen here (note that ultrarefraction is defined in the photonics literature as angles of refraction $\theta_c \simeq 0$ [93]).

We also construct the band surfaces for these same geometries in Figure 5.17, but they are now scaled to the area of circle of radius $a = 0.1$. Here the band surfaces for scatterers with four and three fold symmetry now bear similarity to the first band surface associated with square arrays of pins, as seen in Smith et al. [136]. That is, in Figure 5.17(a) we have absolute maxima at $(\kappa_x, \kappa_y) = (\pm \pi, \pm \pi)$ of $k = 4.7211$, absolute minima at $(\kappa_x, \kappa_y) = (0, \pm \pi)$ and $(\kappa_x, \kappa_y) = (0, \pm \pi)$ of $k = 4.5511$ and a relative maxima at the origin of $k = 4.5643$. Similarly in Figure 5.17(b) the band surface has an absolute maximum of $k = 4.7538$, minimum of $k = 4.5670$ and relative maximum of $k = 4.5868$ at these same coordinates and in Figure 5.17(c) we have an absolute maximum of $k = 4.7224$, minimum of $k = 4.6295$ and relative maximum of $k = 4.6597$ at these points. For Figure 5.17(d) associated with the three leaf clover geometry we have a clear break in up-down symmetry, but still retain relative maxima at the extreme edge of the Brillouin zone of $k = 4.7572$, relative minima at $(0, \pm \pi)$ and $(\pm \pi, 0)$ of $k = 4.6194$ and an relative maximum at the origin of $k = 4.6379$.

For the ellipse in Figure 5.17(e) and the diamond in Figure 5.17(f) there is still a pronounced difference in band surface curvature compared with the more compact geometries, except that we now have relative maxima in the extreme edges of the Brillouin zone. For the case of the ellipse, we still see horizontal propagation as the preferred direction for values below the central saddle point (i.e., $k = 4.6150$) and vertical propagation as the preferred direction for values above $k = 4.6150$ with $\theta_c \simeq 0$, i.e. ultrarefractive properties. In total we have four relative maxima at $(\pm \pi, \pm \pi)$ with $k = 4.7538$ and two minima at $(0, \pm \pi)$ for $k = 4.4800$. There are also two saddle points at $(\pm \pi, 0)$. A similar picture is seen for the diamond scatterer with absolute maximum values of $k = 4.7191$, minimum values of $k = 4.4946$ and a central saddle point value of $k = 4.6015$.

Consequently it would seem that band surfaces for scatterers with large aspect ratios (that is, long and slender scatterers) have very different isofrequency contours compared to those of more compact geometry, acting as platonic polarisers and ultrarefractive media. Another interesting feature for all of the geometries considered is the presence of a stop band below the first band surface (at low $k$) which persists even as we shrink the area of the scatterer down to zero (i.e., consider a pinned inclusion) [106].
Figure 5.17: Isofrequency contours of the first band surface for the geometry: (a) circle \((a = 0.1)\), (b) L4 shape, (c) four-leaf clover, (d) three-leaf clover, (e) ellipse, and (f) diamond.

In Figure 5.18 we examine vertical slices of the band surfaces as we transition from circular to elliptical scatterers (whilst preserving area) at \(\kappa_x = 0\) and \(\kappa_x = \pi\). Here we see platonic polarisation becoming more pronounced as \(b/a\) is increased. We also see what appears to be an ultraflat band near \(b/a = 1.2218\) and in Figure 5.18(b) we have a minimum range of \(k\) at \(b/a = 1.2414\) which suggests that there is always a degree of curvature present in the band surfaces associated with an ellipse. If we examine the average curvature across these segments, we see a minimum value of 0.0026 at \(b/a = 1.2\) for \(\kappa_x = 0\) and a minimum value of 0.0020 at \(b/a = 1.2323\) for \(\kappa_x = \pi\).
We consequently propose that there exist ultraflat bands between these particular aspect ratios.

In Figure 5.19 we construct the band surfaces for an ellipse \((b/a = 2, a = 0.2)\) when rotated by 30 degrees in Figure 5.19(a) and by 45 degrees in Figure 5.19(b), clockwise about the origin. Under rotation by 30 degrees we see an increase in horizontal propagation preference as opposed the vertical direction, as well as a decrease in the \(k\)-range of the band. The maximum value of \(k = 5.7630\) is achieved at \((0, \pm \pi)\) and the minimum values are at \((\pm \pi, 0)\) for \(k = 5.6472\) with a central saddle value of \(k = 5.7321\). For 45 degree rotation we see an increased degree of isotropy in the centre of the band surface compared to Figure 5.16(e), however, the isofrequency contours are predominantly elliptic throughout. We have a central maximum of \(k = 5.7815\), four minima of \(k = 5.6748\) at the edges of the Brillouin zone and four saddle points.

In Table 5.5 we present a table of \(k\) values corresponding to selected points throughout the Brillouin zone. The values obtained for the circular scatterer show good agreement with the
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Table 5.5: Values of $k$ for the first band surface corresponding to a square array of clamped scatterers. Here Coordinates 1, 2 and 3 refer to $(\kappa_x, \kappa_y) = (\pi/2, 0)$, $(\kappa_x, \kappa_y) = (\pi, \pi/2)$ and $(\kappa_x, \kappa_y) = (\pi/2, \pi/2)$ respectively.

<table>
<thead>
<tr>
<th>Scatterer</th>
<th>Coordinate 1</th>
<th>Coordinate 2</th>
<th>Coordinate 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle ($a = 0.2$)</td>
<td>Present</td>
<td>5.634548</td>
<td>5.590603</td>
</tr>
<tr>
<td></td>
<td>Poulton et al. [124]</td>
<td>5.634877</td>
<td>5.590820</td>
</tr>
<tr>
<td>L4 geometry</td>
<td></td>
<td>5.736497</td>
<td>5.714475</td>
</tr>
<tr>
<td>Four leaf clover</td>
<td></td>
<td>5.719556</td>
<td>5.612017</td>
</tr>
<tr>
<td>Three leaf clover</td>
<td></td>
<td>5.806273</td>
<td>5.751169</td>
</tr>
<tr>
<td>Ellipse ($b/a = 2$)</td>
<td></td>
<td>5.576275</td>
<td>5.585914</td>
</tr>
<tr>
<td>Diamond geometry</td>
<td></td>
<td>5.577697</td>
<td>5.530857</td>
</tr>
</tbody>
</table>

Table 5.6: Values of $k$ for the first band surface corresponding to a square array of clamped scatterers scaled to preserve area of circle radius $a = 0.1$.

<table>
<thead>
<tr>
<th>Scatterer</th>
<th>Coordinate 1</th>
<th>Coordinate 2</th>
<th>Coordinate 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle ($a = 0.1$)</td>
<td>Present</td>
<td>4.557725</td>
<td>4.615795</td>
</tr>
<tr>
<td></td>
<td>Poulton et al. [124]</td>
<td>4.557812</td>
<td>4.615857</td>
</tr>
<tr>
<td>L4 geometry</td>
<td></td>
<td>4.580074</td>
<td>4.643568</td>
</tr>
<tr>
<td>Four leaf clover</td>
<td></td>
<td>4.645573</td>
<td>4.665761</td>
</tr>
<tr>
<td>Three leaf clover</td>
<td></td>
<td>4.629980</td>
<td>4.672998</td>
</tr>
<tr>
<td>Ellipse ($b/a = 2$)</td>
<td></td>
<td>4.552513</td>
<td>4.578742</td>
</tr>
<tr>
<td>Diamond geometry</td>
<td></td>
<td>4.552810</td>
<td>4.577746</td>
</tr>
</tbody>
</table>

values obtained in Poulton et al. [124], and are obtained by interpolation over the scattered data obtained using the procedure outlined in Section 5.5.7. These values are included for reference. A similar table of values is presented in Table 5.6 when the scatterers are scaled to preserve the area associated with a circle of radius $a = 0.1$.

In Figure 5.20 we examine the reflection and transmission curves for a single grating of scatterers. In particular we consider $|R_0|$ and $|T_0|$ for a single array constructed of circles in Figure 5.20(a), an array of three-leaf clovers in Figure 5.20(b) and an array of ellipses ($b/a = 2$) in Figure 5.20(c). We consider these coefficients over the range $\pi/2 < k < 3\pi$ as for $k \leq \pi/2$ we have $|T_0| \simeq 0$ and $|R_0| \simeq 1$ for all three geometries. In all three of these figures we can clearly see the effect of Wood anomalies on the reflection and transmission curves, as they become non-differentiable at these points, creating pronounced dips and spikes in the curves. These Wood anomalies take place at $k = 4.1888$ and $k = 8.3776$ over this interval for the fixed incident angle $\theta_i = \pi/6$ (irrespective of geometry). For a circular grating we have a minimum in reflection of $|R_0| = 0.2156$ at $k = 7.5335$ and a minimum of transmission ($|T_0| \simeq 0$) at $k = 5.9228$. We have an interval of minimal transmission over $5.5677 < k < 5.6554$ which is associated with the first band surface for the doubly periodic array problem, and this is reflected in the array problem by group velocity vectors of small magnitude. For the three leaf clover array in Figure 5.20(b) we have a minimum value of reflection $|R_0| = 0.2059$ at $k = 7.7174$ and a minimum in transmission at $k = 6.2515$. For the elliptical grating we see stronger transmission compared to the other
two curves which is characterised by a zero in transmission at $k = 5.0802$ and a maximum in transmission of $|T_0| = 0.4518$ at $k = 7.1934$, along with an earlier minimum value in reflection of $|R_0| = 0.1097$ at $k = 6.4118$. 

Figure 5.20: Reflection and Transmission curves for a single grating constructed of (a) circles of radius $a = 0.2$, (b) three leaf clovers, (c) ellipses ($b/a = 2$) for a Helmholtz incident wave at $\theta_i = \pi/6$. 

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5.5 Publication: “Flexural wave filtering and platonic polarisers in thin elastic plates”

5.5.9 Conclusions

In this paper we have demonstrated how to construct the solution for both a grating and doubly periodic square array comprised of scatterers which are of arbitrary geometry. The solution is constructed using boundary integral equations, quasiperiodic Green’s functions and scattering matrices, which allow us to find band surfaces for geometries of arbitrary shape. We have calculated the first band surfaces for a select number of geometries which have been shown to exhibit a number of complex scattering behaviours including negative refraction, preferential directions (platonic polarisation), stop bands at low wave number, ultrarerefraction and anisotropic wave propagation. There have also been a number of interesting behaviours associated with elliptical geometries, including induced isotropy under rotation and the existence of ultraflat bands. Future work includes the construction of band surfaces with free-edge boundary conditions at the edge, and an extension of the boundary integral theory outlined here to arrays of cracks of arbitrary path, in a related vein to work by Porter and Evans [119, 120].

Acknowledgements

M.J.A.S. would like to acknowledge John D’Errico for the program interparc.m which ensures approximately equidistant discretisation of our scattering geometries\textsuperscript{1}.

5.5.10 Appendix A: Band surface calculation for circular cavities

For the particular case of an array of circular cavities one can represent the solution to (S.1) in terms of multipoles. This solution approach is outlined in Movchan et al. [106], Poulton et al. [124] and is included here as a check on numerics. Inside the fundamental Wigner–Seitz cell we can expand the homogeneous solution as

\[
    w(x) = \sum_{n=-\infty}^{\infty} \left( A_n J_n (kr) + B_n I_n (kr) + E_n H_n^{(1)} (kr) + F_n K_n (kr) \right) e^{in\theta}.
\] (S.48)

After imposing the clamped-edge conditions (S.8) at \( r = a \), we obtain the system

\[
    A_n J_n (ka) + B_n I_n (ka) + E_n H_n^{(1)} (ka) + F_n K_n (ka) = 0,
\] (S.49)

\[
    A_n J'_n (ka) + B_n I'_n (ka) + E_n H_n^{(1)'} (ka) + F_n K'_n (ka) = 0,
\] (S.50)

\textsuperscript{1}Available at: www.mathworks.com
where prime notation denotes derivatives with respect to \( r \). This can then be solved after substitution of the appropriate dynamic Rayleigh identities:

\[
A_n = \sum_{l=-\infty}^{\infty} (-1)^{l-n} s_{l-n}^{H,A} E_l, \quad \text{and} \quad B_n = \sum_{l=-\infty}^{\infty} (-1)^l s_{l-n}^{K,A} F_l. \tag{S.51}
\]

Here \( s_{l}^{H,A} \) and \( s_{l}^{K,A} \) are lattice (array) sums as opposed to grating sums, which are given in convergent form by

\[
s_{m}^{H,A}(k, \kappa) = -\delta_{m0} + i s_{m}^{Y,A}(k, \kappa), \tag{S.52}
\]

and

\[
s_{m}^{Y,A}(k, \kappa) = \frac{1}{J_{m+3}(k \xi)} \left( -\left[ Y_{3}(k \xi) + \frac{3}{\pi} \sum_{n=1}^{3} (2-n)! \left( \frac{2}{k \xi} \right)^{3-2n+2} \right] \delta_{m0} \right.
\]

\[
-\frac{4}{d^2} \sum_{p} \left( k/Q_p \right)^3 \frac{J_{m+3}(Q_p \xi) e^{im \theta_p}}{Q_p^2 + k^2}, \tag{S.53}
\]

and

\[
s_{m}^{K,A}(k, \kappa) = \frac{1}{J_{m+3}(k \xi)} \left( \left[ K_{3}(k \xi) - \frac{8}{(k \xi)^3} + \frac{1}{k \xi} - \frac{k \xi}{8} \right] \delta_{m0} \right.
\]

\[
+\frac{2\pi}{d^2} \sum_{p} \left( k/Q_p \right)^3 \frac{J_{m+3}(Q_p \xi) e^{im \theta_p}}{Q_p^2 + k^2}, \tag{S.54}
\]

where \( \delta_{nn} \) is the Kronecker delta, \( \theta_p = \arg Q_p, Q_p = (\kappa_x + 2\pi n/d, \kappa_y + 2\pi m/d), Q_p = \|Q_p\|_2, \) and the vector \( \xi, \) with corresponding norm \( \xi, \) represents an arbitrary vector positioned inside the first Brillouin zone. Here we have used an acceleration parameter of 3, which is within the recommended range specified in Movchan et al. [106].

The lattice sum expressions above are valid for square arrays and \( m \geq 0, \) and so for \( m < 0 \) one can use the identities \( s_{-m}^{Y,A}(k, \kappa) = \left[ s_{m}^{Y,A}(k, \kappa) \right]^* \) and \( s_{-m}^{K,A}(k, \kappa) = (-1)^m \left[ s_{m}^{K,A}(k, \kappa) \right]^*, \) as outlined in [25].

Substituting the dynamic Rayleigh identities into the system above allows us to obtain

\[
\sum_{l=-\infty}^{\infty} \left[ (-1)^{l-n} s_{l-n}^{H,A} J_n(ka) + \delta_{ln} H^{(1)}_n(ka) \right] E_l + \sum_{l=-\infty}^{\infty} \left[ (-1)^l s_{l-n}^{K,A} I_n(ka) + \delta_{ln} K_n(ka) \right] F_l = 0, \tag{S.55}
\]
and

\[
\begin{align*}
\sum_{l=-\infty}^{\infty} \left[ (-1)^{l-n} S_{l-n}^{HA} J_n'(ka) + \delta_{ln} H_n^{(1)'}(ka) \right] E_l \\
+ \sum_{l=-\infty}^{\infty} \left[ (-1)^{l} S_{l-n}^{KA} I_n'(ka) + \delta_{ln} K_n'(ka) \right] F_l &= 0,
\end{align*}
\]  
(S.56)

which can be solved after suitable truncation at \( l = \pm N \) and \( n = \pm N \), admitting the block matrix system

\[
\mathbf{A}(k, \kappa) \begin{pmatrix} \mathbf{E} \\ \mathbf{F} \end{pmatrix} = 0.
\]  
(S.57)

Here \( \mathbf{E} \) and \( \mathbf{F} \) represent vectors containing the unknown \( E_l \) and \( F_l \) coefficients and \( \mathbf{A} \) is a block matrix of dimension \((4N+2) \times (4N+2)\). To construct the band surface we search for values of \( k \) such that \( \det(\mathbf{A}) = 0 \), for fixed \( \kappa \).
5.6 Discussion

In this chapter we have provided a solution outline for the problem of wave scattering by a single inclusion of arbitrary geometry, as well as one- and two-dimensional square arrays of arbitrary scatterers in a thin elastic plate. This was achieved by forming and solving an appropriate system of boundary integral equations after imposing boundary conditions at the cavity edge and was outlined in the papers “Scattering by cavities of arbitrary shape in an infinite plate and associated vibration problems” and “Flexural wave filtering and platonic polarisers in thin elastic plates”. For the array problem we considered only clamped-edge conditions due to the complexity of the solution method for free-edge and other boundary conditions. Several key expressions, as well as the expressions for the boundary conditions, become intractable for complex boundary conditions such as free-edge or simply-supported conditions (and would also require Fourier series expansion to solve), and consequently a modified, or even alternative, approach is suggested for non-clamped boundary conditions.
Concluding remarks

The research outlined in this thesis has dealt with the behaviour of flexural waves in structured elastic plates, the study of which is known as platonics. This work has contributed to the understanding of wave propagation through two-dimensional arrays, as well as wave scattering by single bodies, gratings and clusters in thin elastic plates, which are governed by the biharmonic plate equation. In the preceding chapters we have provided a solution outline for various platonic structures comprised of circular, pinned, and arbitrarily shaped inclusions. We have also demonstrated that plates can guide and disperse flexural waves analogously to the way photonic crystals guide and disperse light. Prior to the publication of the papers which constitute this thesis, an understanding of only certain geometries was known (in particular, simple configurations of circular and pinned scatterers), however we have extended this to arbitrary shapes and contributed to the understanding of defective pinned PlaCs. We have also been able to confirm the anisotropic behaviour of pinned PlaCs using Gaussian beams, and have been able to demonstrate that these structures can support a number of complex diffraction behaviours.

An understanding of wave behaviour through structured plates is highly important as it allows us to control potentially destructive wave energy which can be converted into energy that can be guided, trapped or harnessed for constructive, or useful purposes. This is particularly relevant for modelling floating elastic plates, and in vibration control [35, 38]. In fact, the strong energy localisation which is seen in waveguides of pinned plates may prove useful for the transport of wave energy over long distances. We have also demonstrated that it is possible to achieve strong energy trapping within defects of pinned PlaCs. Interesting prospects for future study include...
the fabrication of complex PlaC designs such as heterostructures (i.e., a combination of different lattice geometries) to achieve high quality factor wave trapping. Quality factors ($Q$-factors) are a measure of how well energy is localised to a structure, and there is ongoing research into high $Q$-factor structures in both the photonic and platonic literature. This includes recent work by Akahane et al. [2], Haslinger et al. [53, 54] and Movchan et al. [107]. In the platonic setting this includes grating pairs of pinned points which have exhibited $Q$-factors of 5400, stacks of three gratings (triplets) with $Q$-factors of $3.6 \times 10^4$ [53] and a novel triplet design (comprised of a central grating of circular inclusions surrounded above and below by a grating of pins) which gives $Q$-factors of $6.54 \times 10^4$ [54]. In the photonic literature, the existence of cavities which have ultrahigh $Q$-factors in the order of $2 \times 10^7$ [141] have been demonstrated, and consequently there is ongoing attention directed at the design of high $Q$-factor structures in platonics.

In this emerging field, there are considerable opportunities to investigate the dynamics of complex structures in elastic plates. To date, only simple shapes and simple array geometries have been examined, and have acted primarily as a proof of concept that PlaCs can demonstrate interesting and unusual diffraction behaviours. Here we have provided a broad framework for aspiring researchers to investigate the dispersive properties of PlaCs, and we have removed the restriction of circular and pinned geometries, so that future researchers can consider any smooth polygon geometry. This is particularly topical as we have shown that scatterers with large aspect ratios can give rise to very exciting wave behaviours for PlaCs in guiding and dispersing flexural energy. Additionally we have provided a solution using boundary integral equations and boundary element methods (BEMs) which has several key advantages over finite element methods (FEMs) particularly as the construction of meshes for complicated polygons can be a time consuming exercise. Furthermore, the arbitrary shape solution provided here for arrays could also be useful in addressing the open question of whether researchers should alter the geometry of a scatterer for a fixed array design, or whether one should alter the geometry of the array for a fixed shape, in order to achieve a desired band structure or curvature. Preliminary work by the author on pinned rectangular arrays (of aspect ratio 2) has shown that the first band surface exhibits near-identical curvature to a square array of clamped elliptical scatterers (of aspect ratio $b/a = 2$).

Additional future work includes investigations into different lattice designs, such as hexagonal and oblique lattices, for any smooth scatterer geometry. To the best of the author’s knowledge, existing designs of PlaCs have been restricted to square and rectangular arrays of circles, squares, point masses and pins, and there has been limited investigation into other Bravais lattice geometries in two dimensions. Similarly there has been some attention paid to scatterers subject to free and clamped boundary conditions at the edge, but little attention paid to arbitrary shapes with simply supported boundary conditions. That said, considerable attention has been paid directed at arrays of simple supports in plates and beams due to applications in aerospace engineering [100, 102, 130]. An extension to three-dimensional platonic structures could also be made, for example by considering three-dimensional arrays of masses connected with Euler–Bernoulli beams, following from the work of Colquitt et al. [27] in two dimensions.
One could also investigate sandwich structures of periodically connected plates, following from Kouzov and Lukyanov [70]. An investigation into the reflection and transmission spectra for finite grating stacks of arbitrary inclusions could also be an easy application of the present framework, especially since the coefficient expressions for $R$ and $T$ that are given in Chapter 5 can be used for any platonic grating (i.e., irrespective of the boundary condition imposed at the edge), and the recurrence relation for multiple stacks of gratings is well known [107].

Other interesting diffraction phenomena could be observed by considering multiple inclusions within a fundamental cell, such as several circular inclusions of different radius. This macro-cell containing several geometries can then be periodically repeated in one or two dimensions to form the desired structure. An investigation into such designs has been considered in the related field of phononic crystals [118], but the author is unaware of any such work in platonics. Note that phononic crystals model waves in an isotropic elastic medium (such as rubber) and are accordingly governed by the Navier-Stokes equation as opposed to the biharmonic plate equation for PlaCs. Furthermore, phononic crystals can support both shear and dilatational waves in an elastic medium, whereas PlaCs feature only flexural waves (of Helmholtz and modified Helmholtz type, i.e., propagating or evanescent waves).

Platonic crystals are not merely confined to the theoretical regime, in fact the first experimental work on the control of flexural waves was performed recently and was concerned with the cloaking of a single clamped circular cavity [145]. By varying the ratio of two materials over an enclosing circular ring design (which is similar to the water wave cloak design see in Farhat et al. [40]), Stenger et al. [145] were able to achieve flexural wave cloaking over a wide frequency range. The importance of this work is highlighted in the review article by McPhedran and Movchan [95] which emphasizes the ongoing work in cloaking in virtually any medium which can support propagating waves. Such investigations in platonics may pave the way for new frontiers in metamaterials research.

An interesting extension to existing PlaC designs could be made by suspending structured elastic plates above a fluid or acoustic medium, following the word of Bennetts and Williams [12] for a single inclusion of arbitrary shape above a body of water of finite depth as well as Evans and Porter [38], Kouzov and Lukyanov [70] for pinned floating elastic plates. This has applications in the modelling of large ice sheets which can contain multiple polynya and in certain circumstances may be modelled as a plate containing an array of such inclusions. In this way, the boundary integral equation systems outlined here could also be modified to deal with the case when the inclusion becomes a crack of arbitrary path. Proposals for other interesting research directions includes analogues to any photonic, phononic and plasmonic structure imaginable, from channel-drop filters (i.e. a mixture of line and point defects) to T-splitter waveguide designs [57]. It would also be interesting to consider half-space problems in platonics. For example, one could investigate the floating half plate problem, where the plate could then be structured to introduce wide band gaps that would reflect wave energy, as well as half crystals (i.e. a semi infinite stack of gratings in a plate of infinite extent) and quarter crystals (i.e., a semi infinite stack of half
gratings). Another next step in PlaCs could be the extension of biharmonic plate structures to Mindlin or Timoshenko plate models (following from Movchan et al. [108]), or possibly even simple elastic shells.

In summary, there are several promising and interesting lines of research for platonics, which is a field still in its infancy. So far we have provided a solution outline to an arbitrary geometry in a two-dimensional array and directly considered a finite of geometries and lattice designs. A thorough investigation of optimal scatterer design or array geometry could be easily be performed using our framework, in order to achieve a desired diffraction behaviour. This is in addition to the multitude of future research directions suggested above.
Appendix A: Evaluation of Grating Sums

In this Appendix we consider the efficient evaluation of grating sums which were obtained earlier during the derivation of the quasiperiodic Green’s functions for the grating problem. In their most basic form these expressions are only conditionally convergent, and require acceleration. This problem has a long history in scattering theory and is addressed in a series of publications from Twersky [153] to Nicorovici and McPhedran [109] and Linton [77]. We provide a brief outline of the procedures and results here for completeness.

The grating sums (2.3.6a) are part of a wider family of series known as Schlomilch series which take the form

\[
\sum_{n=1}^{\infty} Z_{2l}(nkd) \cos(n\alpha_0 d), \quad (A.1a)
\]

\[
\sum_{n=1}^{\infty} Z_{2l+1}(nkd) \sin(n\alpha_0 d), \quad (A.1b)
\]

where \(Z_n(z)\) is an arbitrary Bessel function. Providing that parameter values are not in the vicinity of a Wood anomaly, it is possible to obtain directly convergent expressions for these series.

Recall that from (2.3.6a) we have

\[
S_l^{H,G}(\alpha_0, k, d) = \sum_{n \neq 0} H_1^{(1)}(|n|kd)e^{i\alpha_0 nd}e^{il\phi_n}
\]

\[
= \sum_{n=\infty}^{-1} H_1^{(1)}(|n|kd)e^{i\alpha_0 nd}e^{il\pi} + \sum_{n=1}^{\infty} H_1^{(1)}(|n|kd)e^{i\alpha_0 nd}
\]

\[
= \sum_{n=1}^{\infty} H_1^{(1)}(|n|kd) \left( e^{i\alpha_0 nd} + (-1)^l e^{-i\alpha_0 nd} \right), \quad (A.2)
\]

and so for consistency of notation with Twersky [153] we consider the sum

\[
\mathcal{H}_l = S_l^{H,G}(-\alpha_0, k, d) = (-1)^l S_l^{H,G}(\alpha_0, k, d). \quad (A.3)
\]
In general, the convergence of Schlomilch series such as \( H_l \) are especially slow, and acceleration by some scheme is necessary for efficient numerical evaluation. Twersky [153] begins by considering the more general form

\[
h_l = \sum_{n=-\infty}^{\infty} \left\{ e^{in\alpha d} H^{(1)}_l (r_n) e^{-il\gamma_m} - H^{(1)}_l (r) i^n e^{-il\theta_0} \right\},
\]

where \( r_m = \sqrt{x^2 + (y-md)^2} \), \( r = \sqrt{x^2 + y^2} \), \( \gamma_m = \tan^{-1}\{ (y-md)/x \} \). This expression is obtained by adding and subtracting an expression involving a \( H^{(1)}_l (z) \) term so that the series \( h_l \) is regular in the limit \( r \to 0 \) and that

\[
\mathcal{H}_l = \lim_{r \to 0} h_l (r).
\]

Using the integral representation for a Hankel function of order \( l \)

\[
H^{(1)}_l (r) i^n e^{-il\phi_n} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i(y-nkd)+ix\chi(t)} e^{il\sin^{-1}(t)} \chi(t) dt,
\]

which is valid for \( x > 0 \), where

\[
\chi(t) = \begin{cases} \sqrt{1-t^2}, & \text{if } |t| \leq 1, \\ i\sqrt{t^2-1}, & \text{if } |t| > 1 \end{cases}
\]

and using Poisson’s summation formula

\[
\sum_{n=-\infty}^{\infty} f(2m\pi + a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{ina} \int_{-\infty}^{\infty} f(t) e^{ist} dt,
\]

then the first term of (A.4) takes the form

\[
\sum_{n=-\infty}^{\infty} e^{in\alpha d} H^{(1)}_l (r_n) e^{-il\gamma_m} = 2 \sum_{m=-\infty}^{\infty} C_m e^{-il\theta_m + iy\sin\theta_m + ix\cos\theta_m}.
\]

This expression is valid for \( x > 0 \) where \( C_m = 1/(kd\chi(\sin\theta_m)) \) and \( \sin\theta_m = \sin\theta_0 + 2m\pi/kd \), for \( m \in Z \). The analogous expression for \( x > 0 \) can be obtained simply by specifying \( \theta_m = \pi - \theta_m \) in (A.8).

For the second term in (A.4) we use the integral representation

\[
H^{(1)}_l (r) i^n e^{-il\phi_0} = 2 \int_{-\infty}^{\infty} C_m e^{-il\theta_m + iy\sin\theta_m + ix\cos\theta_m} dm,
\]

which is valid for \( x > 0 \) and consequently

\[
h_l (r) = 2 \left\{ \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} dm \right\} C_m e^{-il\theta_m + iy\sin\theta_m + ix\cos\theta_m}.
\]
Note that the analogous expression for \( x < 0 \) can be obtained by specifying \( \theta_m = \pi - \theta_m \) in the above. Returning to the limit definition in (A.5) and specifying \( y = 0 \) and \( x = \epsilon \) we have

\[
H_l = \lim_{r \to 0} h_l(r)
= \lim_{\epsilon \to 0} \left\{ \sum_{m = -\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dm}{\alpha} \right\} C_m e^{-il\theta_m + iy\sin \theta_m + i\epsilon \cos \theta_m}
= \lim_{\epsilon \to 0} 2SC_m e^{-il\theta_m},
\]

where we define the operator

\[
S = \left\{ \sum_{m = -\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dm}{\alpha} \right\}.
\]

An identical procedure for \( x < 0 \) reveals

\[
H_l = 2SC_m e^{-il(\pi - \theta_m)}.
\]

If one takes the mean of these two equivalent expressions it is then possible to obtain

\[
H_l = SC_m \left[ e^{-il\theta_m} + e^{-il(\pi - \theta_m)} \right],
\]

which satisfies the relation \( H_{-l} = (-1)^l H_l \) and so only \( l > 0 \) terms are considered here. We decompose \( S = S_+ + S_- \) where \( S_+ \) takes advantage of the representation

\[
\lim_{\epsilon \to 0} \left\{ \int_{0}^{\infty} z^{-1} rme^{-zr} \frac{dz}{\alpha} - \sum_{n = 0}^{\infty} \frac{(n + x)^{-1}e^{-(n+x)\epsilon}}{l} \right\} = \frac{B_l(x)}{l},
\]

and \( B_l(x) \) represents a Bernoulli polynomial. This result was first shown by Nörlund [110] and a generalisation of this limit is considered also Thompson and Linton [150]. Consequently one obtains the representation

\[
H_l = 2S_- \left\{ C_m e^{-il(\pi - \theta_m)} \right\} + 2S_+ \left\{ C_m e^{-il(\pi - \theta_m)} \right\},
\]

where

\[
S_- f(m) = \lim_{\epsilon \to 0} \left( \sum_{m = -\infty}^{-1} - \int_{-\infty}^{-\alpha_0l/2\pi} \frac{dm}{\alpha} \right) f(m) e^{i\epsilon \cos \theta_m},
\]

\[
S_+ f(m) = \lim_{\epsilon \to 0} \left( \sum_{m = 0}^{\infty} - \int_{-\infty}^{\alpha_0l/2\pi} \frac{dm}{\alpha} \right) f(m) e^{i\epsilon \cos \theta_m}.
\]

We can further decompose \( S_+ \) by adding and subtracting \( e^{-il\theta_m} \) to obtain

\[
H_l = 2S_- \left\{ C_m e^{-il(\pi - \theta_m)} \right\} + 2S_+ \left\{ C_m e^{-il\theta_m} \right\} + F_l,
\]
where

\[ F_l = 2S_+ C_m \left[ e^{-i\pi(\pi_m - \theta_m)} - e^{-i\theta_m} \right]. \]  

(A.19)

Using the integral relation

\[ 2 \int_{-\infty}^{-\alpha_0 d/2\pi} C_m e^{-il\theta_m} dm = 2 \int_{-\infty}^{\infty} C_m e^{il\theta_m} dm = \frac{1}{\pi} \int_{-\pi/2+i\infty}^{\infty} \frac{e^{in\theta} d\theta}{\pi}. \]  

(A.20)

one can represent (A.18) in the form

\[ \mathcal{H}_l = e^{-i\pi} \left[ 2 \sum_{m=-\infty}^{1} C_m e^{il\theta_m} - \frac{1}{il\pi} \right] + \left[ 2 \sum_{m=1}^{\infty} C_m e^{-il\theta_m} - \frac{1}{il\pi} \right] + F_l. \]  

(A.21)

The final difficulty that remains is the evaluation of \( F_l \). When \( l \) is even we have

\[ \sin(2l\theta) = \sum_{m=1}^{l} (-1)^{m-1} 2^{m-1} (l + m - 1)! \frac{\sin \theta}{(2m-1)!(l-m)!}, \]  

(A.22)

where we make use of the Bernoulli polynomial representation (A.15) to obtain

\[ F_{2l} = \frac{4i}{kd} \sum_{m=1}^{l} \frac{(-1)^{m-1} 2^{m-1} (l + m - 1)! B_{2m}(\alpha_0 d/2\pi)}{(2m-1)!(l-m)!} \frac{k\theta}{(2\pi)^{2m-1}2m}. \]  

(A.23)

When \( l \) is odd we make use of the relation

\[ \cos([2l+1]\theta) = \sum_{m=0}^{l} \frac{(-1)^m 2^m (n + m)!}{(2m)!(n-m)!}, \]  

(A.24)

and the Bernoulli polynomial representation (A.15) to obtain

\[ F_{2l+1} = \frac{4}{kd} \sum_{m=0}^{l} \frac{(-1)^m 2^m (l + m)! B_{2m+1}(\alpha_0 d/2\pi)}{(2m)!(l-m)!} \frac{k\theta}{(2\pi)^{2m}2m + 1}. \]  

(A.25)

Consequently we can obtain the elementary function representations

\[ \mathcal{H}_{2l} = 2 \sum_{n=1}^{\infty} H_{2l}(nk) \cos(nk\alpha_0 d)e^{-i\pi} \]

\[ = 2 \sum_{m=0}^{\infty} C_m e^{-2i\theta_m} + 2 \sum_{m=-\infty}^{1} C_m e^{2i\theta_m} + \frac{i}{l\pi} \sum_{m=1}^{l} \frac{(-1)^{m-1} 2^{m-1} (l + m - 1)! B_{2m}(\alpha_0 d/2\pi)}{(2m-1)!(l-m)!} \frac{k\theta}{(2\pi)^{2m-1}2m}. \]  

(A.26)
and

\[ \mathcal{K}_{2l+1} = -2i \sum_{n=1}^{\infty} H_{2l+1}(nk) \sin(nk\alpha_0 d) e^{-il\pi} \]

\[ = 2 \sum_{m=0}^{\infty} C_m e^{-i(2l+1)\theta_m} - 2 \sum_{m=-\infty}^{-1} C_m e^{i(2l+1)\theta_m} + \frac{2}{\pi} + \frac{i}{\pi} \sum_{m=0}^{l} \frac{(-1)^m 2^m (l + m)!}{(2m)! (l - m)!} \frac{B_{2m+1}(\alpha_0 d/2\pi)}{(kd/2\pi)^{2m}(2m+1)}, \]  

(A.27)

where \( S_{2l}^{H,G} = \mathcal{K}_{2l} \) and \( S_{2l+1}^{H,G} = -\mathcal{K}_{2l+1} \).

The expressions above are directly convergent, however one may apply Kummer’s transformation to these (as in Linton [77]) to obtain the expressions (S.17a), (S.17b) and (S.17c).
Appendix B: Evaluation of Lattice Sums

Here we aim to derive convergent expressions for the lattice sums which feature in our two-dimensional PlaCs. These arise from the use of Graf’s addition theorem [1] on the associated quasiperiodic Green’s functions for the lattice. We provide an outline based on the work of Movchan et al. [105], Poulton et al. [121] and Chin et al. [25] for a square array of period \( d \).

The procedure here begins by by obtaining the spectral representations of our Green’s functions (which are also known as the reciprocal space, or plane wave representations). This is done by expressing the Green’s function for the Helmholtz equation (2.4.4a) as the following ansatz

\[
G_a^H(x, x') = \sum_h \tilde{g}(Q_h) e^{iQ_h \cdot \xi},
\]

(B.1)

where \( Q_h = K_h + \kappa \) with norm \( Q_h \), \( \xi = x - x' \) with norm \( \xi \), and \( \kappa \) is an arbitrary Bloch vector with corresponding length \( k \). Here we define the reciprocal lattice vector as

\[
K_h = \frac{2\pi}{d} (m, n), \quad \text{where } h = (m, n) \in \mathbb{Z}_2,
\]

(B.2)

for a square lattice of period \( d \).

Substituting (B.1) into the Helmholtz equation we obtain

\[
\sum_h (-Q_h^2 + k^2) \tilde{g}(Q_h) e^{iQ_h \cdot \xi} = -\sum_p \delta(\xi - R_p) e^{i\kappa \cdot R_p},
\]

(B.3)

where one can use Poisson’s summation formula and the Jacobi-Anger identity

\[
\frac{1}{A} \sum_h e^{iQ_h \cdot \xi} = \sum_p \delta(\xi - R_p) e^{i\kappa \cdot R_p},
\]

(B.4a)

\[
e^{iQ_h \cdot \xi} = \sum_{l=-\infty}^{\infty} i^l J_l(Q_h \xi) e^{i[l(Q_h \cdot \gamma - \gamma)],}
\]

(B.4b)
respectively, to obtain the spectral representation

\[ G^H_a(x, x') = \frac{1}{A} \sum_{h} e^{iQ_h \cdot \xi} = \frac{1}{A} \sum_{l=\infty}^{\infty} \sum_{h} J_l(Q_h \xi) e^{il(\Theta_h - \gamma)} , \]  

(B.5)

where \( \Theta_h = \arg Q_h \) and \( \gamma = \arg \xi \), and \( A \) denotes the volume of a parallelogram formed by the basis vectors of \( K_h \) (1.2.8). In two dimensions this is identical to the area of the unit cell (i.e., \( A = d^2 \) for a square array of period \( d \)), but in three dimensions \( A \) denotes the volume of a parallelepiped formed by the basis vectors of \( K_h \).

Equating (2.4.4a) and (B.5) one obtains the expression

\[ S^H_A l (k, \kappa) J_l (k \xi) = -H^{(1)}_0(k \xi) \delta_{l0} - \frac{4}{A} i^{l+1} \sum_{h} J_l(Q_h \xi) e^{il(\Theta_h)} , \]  

(B.6)

however we can simplify this further by considering the definition of the Hankel function (i.e., \( H^{(1)}_n(z) = J_n(z) + iY_n(z) \)), and so \( S^H_A l (k, \kappa) = S^J_A l (k, \kappa) + iS^Y_A l (k, \kappa) \)). We now consider the spectral representation of the Poisson summation formula

\[ \left( \frac{2\pi}{A} \right)^2 \sum_{h} \delta(\kappa_s - \kappa - K_h) = \sum_{p} e^{i(\kappa - \kappa_s) \cdot R_p} , \]  

(B.7)

where \( \kappa_s \) is taken to be a vector of length \( k \) and argument \( \vartheta_s \) with the property \( \kappa_s - \kappa - K_h \neq 0 \). Subsequently, this relation can subsequently be expressed in the form

\[ \sum_{p} e^{i(\kappa - \kappa_s) \cdot R_p} = 0 . \]  

(B.8)

Substituting the Jacobi-Anger identity

\[ e^{-i\kappa_s \cdot R_p} = \sum_{l=-\infty}^{\infty} (-i)^l J_l(k R_p) e^{il(\Theta_p - \vartheta_s)} , \]  

(B.9)

into (B.8) after isolating the \( p = (0, 0) \) term one obtains

\[ \sum_{l=-\infty}^{\infty} S^J_A l (k, \kappa) e^{-il \vartheta_s} = -1 , \]  

(B.10)

where \( S^J_A l \) is defined previously at (2.4.6a). Integrating both sides of this expression with respect to \( \vartheta_s \) between \(-\pi\) to \( \pi\) one obtains

\[ S^J_A l (k, \kappa) = -\delta_{l0} , \]  

(B.11)

where \( \delta_{mn} \) denotes the Kronecker delta function, and consequently (B.6) takes the form

\[ S^Y_A l (k, \kappa) J_l (k \xi) = -Y_0(k \xi) \delta_{l0} - \frac{4}{A} i^{l+1} \sum_{h} J_l(Q_h \xi) e^{il(\Theta_h)} . \]  

(B.12)
We can accelerate the convergence of (B.12) by multiplying the entire expression by $\xi^{l+1}$ and integrating from 0 to $\eta$, where $\eta$ is an arbitrary parameter with the property $\eta < 1$. At each integration step we then specify $\eta = \xi$, multiply both sides by $\xi$ and make use of the recurrence relations

$$\int_0^\eta \xi^{l+1} J_l(k\xi) d\xi = \eta^{l+1} \frac{J_{l+1}(k\eta)}{k}, \quad \text{(B.13a)}$$

$$\int_0^\eta \xi^{l+1} Y_l(k\xi) d\xi = \eta^{l+1} \frac{Y_{l+1}(k\eta)}{k} + \frac{2^{l+1} l!}{\pi k^{l+2}}, \quad \text{(B.13b)}$$

[1] to reveal the convergent expressions

$$S_m^{Y,A}(k, \kappa) = \frac{1}{J_{m+p}(k\eta)} \left[ \left( \frac{2}{k} \right)^{p-2n+2} \sum_{n=1}^p \left( \frac{p - n}{(n-1)!} \right) \frac{J_{m+p}(Q_h \eta) e^{im\theta_h}}{Q_h^2 + k^2} \right] \delta_{m,0}$$

$$- \frac{4}{A^2} \sum_h \left( \frac{k}{Q_h} \right)^p \frac{J_{m+p}(Q_h \eta)}{Q_h^2 + k^2} e^{im\theta_h}, \quad \text{(B.14)}$$

where $p$ represents the number of integration steps performed. Typically values between $p = 3$ and $p = 7$ are chosen, but for $k \ll 1$ high $p$ values may give rise to numerical instability [105]. By performing the integration step the parameter $\eta$ now denotes an arbitrary vector contained within the first Brillouin zone. An identical procedure can be used for the array sum corresponding to the modified Helmholtz equation to obtain

$$G_a^M(x, x') = \frac{1}{A} \sum_h \frac{e^{iQ_h \xi}}{Q_h^2 + k^2} = \frac{1}{A} \sum_{l=-\infty}^{\infty} \sum_h \frac{i^l J_l(Q_h \xi) e^{il(\theta_h - \gamma)}}{Q_h^2 + k^2}, \quad \text{(B.15)}$$

along with the relation

$$S_l^{K,A}(k, \kappa) I_l(k\xi) = -K_0(k\xi) \delta_{l,0} - \frac{2\pi i}{A} \sum_h \frac{J_l(Q_h \xi) e^{il(\theta_h - \gamma)}}{Q_h^2 + k^2}. \quad \text{(B.16)}$$

This can be repeatedly integrated using the recurrence relations

$$\int_0^\eta \xi^{l+1} I_l(k\xi) d\xi = \eta^{l+1} \frac{I_{l+1}(k\eta)}{k}, \quad \text{(B.17a)}$$

$$\int_0^\eta \xi^{l+1} K_l(k\xi) d\xi = \frac{2^l l!}{k^{l+2}} - \eta^{l+1} \frac{K_{l+1}(k\eta)}{k}, \quad \text{(B.17b)}$$

revealing the general expression

$$S_m^{K,A}(k, \kappa) = \frac{1}{I_{m+p}(k\eta)} \left( \frac{2\pi i}{A} \sum_h \left( \frac{k}{Q_h} \right)^p \frac{J_{m+p}(Q_h \eta) e^{il(\theta_h)}}{Q_h^2 + k^2} \right) \sum_{n=1}^p \frac{(-1)^{p+1} K_p(k\eta)}{2} \left( \frac{2}{k\eta} \right)^{p-2n+2} \delta_{m,0}. \quad \text{(B.18)}$$
Note that the array sum $S^K_A(k, \kappa)$ is directly convergent and does not require acceleration, however there may be circumstances where acceleration is required. The lattice sum expressions above are only valid for $m \geq 0$, and so for $m < 0$ one can use the identities $S^Y_A(k, \kappa) = \left[S^Y_A(k, \kappa)\right]^*$ and $S^K_A(k, \kappa) = (-1)^m \left[S^K_A(k, \kappa)\right]^*$, as outlined in [25].
Bibliography


