On the Stability and Postbuckling Behavior of Shells With Corrugated Cross Sections Under External Pressure

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The problem of determining the deformation of a longitudinally corrugated, long cylindrical shell under external pressure is considered. The topics that are covered can be summarized as follows: the formulation of a boundary value problem for the incremental approach as a normal system of differential equations under appropriate boundary conditions, the determination of postbuckling behavior characteristics for cylindrical shells using the discrete orthogonalization method, and an analysis of deformation for both closed and open cylindrical shells. In particular, we consider the stability and postbuckling behavior of both isotropic and composite shells. The solution is based on the relationships for the cubic version of nonlinear Timoshenko-type shell theory. A comparison is made with the well-established quadratic version, as well as analytical solutions where applicable. The necessity for using more precise equations to examine the postbuckling behavior of shells is shown. Using this higher-order approach, it is possible to determine the postbuckling behavior with much greater accuracy.
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1 Introduction

The buckling and postbuckling behaviors of isotropic circular cylindrical shells under external pressure has been studied sufficiently well for both the elastic and plastic states [1,2]. In general, the study of cylindrical shells with a corrugated cross section is limited to shells of medium length under axial compression [3–5], and in the case of uniform surface pressure, preference is usually given to cylindrical shells with a corrugated generator [6–9]. In practical terms, long shells are used to model pipelines. However, for such structures, it is necessary to ensure their stability for the case of bending by edge forces and moments, in addition to their required stability margin under external pressure. Shells with a corrugated generator cannot meet these two requirements, while for shells with a corrugated guide, it is not obvious. In Refs. [10,11], it was shown that there is one type of corrugated shell which can have a higher critical external pressure than circular shells of the same thickness. For this type of corrugation, the cross section of shells are formed by arcs of circles of radius r whose ends are simply supported on a circle of radius $R_0 > r$. These shells have a high longitudinal rigidity that provides strong reliability under bending. An investigation into the stability of long shells with this type of corrugation has not yet been conducted. That is, the stability and postbuckling behavior of long cylindrical shells with a corrugated cross-section under external pressure is considered here. We will investigate not only isotropic elastic membranes, but also consider shells constructed of composites, which are characterized by reduced shear stiffness. The shell theories of Refs. [12,13] that are commonly used in similar problems of this type are not suitable because they only consider this specific property. As such, the first step is to obtain the equations corresponding to Timoshenko-type shell theory, valid for large deflections and angles of rotation. The well-known low-order version of this theory is not appropriate, as it is limited to small deformations and small angles of rotation.

This work begins by evaluating expressions for the deformations and the curvature increment. Particular attention is paid to the function χ that determines the change in the deflection of the shell, using both exact and cubic representations. From these relations, expressions for the cubic Timoshenko-type nonlinear theory and exact analytical expressions for the deformations and curvature are obtained. Based on the principle of possible displacements the nonlinear equilibrium equations for computing the stress-strain state of closed and open cylindrical shells are determined. Construction of the resolving system of equations in incremental form is then considered, which is solved using the discrete orthogonalization method [14]. In the proposed version of the algorithm, the load parameter can be considered as a new resolving function [15], which for this case, remains unchanged during the passage of regular and singular points on the trajectory of equilibrium paths.

In the following section the analytical solution for the problem of determining the stability of cylindrical shells under an external load is shown. The final stage of the work is to conduct a parametric analysis to confirm the hypothesis that we can observe a significant increase in the critical values of external pressure due to the corrugated cross section. The calculation results for the stability and postbuckling behavior for a particular corrugated shell is given in the subsequent section. In the results section, the difference between the cubic and quadratic versions is examined in calculating the value of loads, as is the effect of reduced shear stiffness on the nature of the postbuckling behavior. Thus, the article can point out three major new results: we show previously unknown features of nonlinear deformation in the subcritical and

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postbuckling states, propose a uniform approach for passing regular and singular points on the equilibrium paths, and finally, we develop a method for solving nonlinear problems using Timoshenko-type shell theory at large angles of rotation, taking into account the transverse shear.

2 Basic Relations for Cubic Timoshenko-Type Shell Theory Under Large Angles of Rotation

Let us consider a cylindrical shell of infinite length. We introduce the Cartesian coordinate system (X, Y, Z) and define X as the direction which coincides with the generator of the reduced surface, Y is directed along the guide of the cylinder, and (X, Y, Z) are the right three vectors in three dimensions. If the load and mechanical characteristics do not depend on the length, each cross section is subject to the same conditions. In this case, the stresses and displacements are independent of the coordinate X. We introduce the polar coordinate system (R, φ) for an arbitrary cross section, and if the function $R(\varphi) = R_0\rho(\varphi)$ is given, then the Lame parameter A_2 and curvature $1/R_2$ can be determined by the known formulas of differential geometry. Since we consider the flat deformation of a cylinder, then components v' and w' of the displacement vector are projections on the Y and Z axes, respectively. According to the Timoshenko kinematic hypothesis, the relationship between the displacements and the coordinate Z is used to be linear

$$v'(Z) = v + Z\psi, \quad w'(Z) = w + Z\chi \tag{1}$$

where v and w are displacements of the coordinate surface and ψ and $1+\chi$ are the direction cosines of the coordinate lines in the deformed state.

Substituting the relationships from Eq. (1) into the expression for the deformations, we obtain nonlinear expressions with respect to the coordinate Z [16]. These expressions give us an approximate idea of the distribution of the displacements across the thickness of the shell, and are given in Refs. [16,17].

In the present case, we require only expressions for the deformation of the center line ε_{22}, the deformation elongation of the normal element ε_{33}, the shear strain ε_{23}, and the curvature increment κ_{22}. In Ref. [17], these expressions are defined as

$$\varepsilon_{22} = \varepsilon_2 + \frac{1}{2}\left(\varepsilon_2^2 + \theta_2^2\right), \qquad \varepsilon_{23} = \psi + \theta_2(1+\chi) + \psi\varepsilon_2 \tag{2a}$$

$$\varepsilon_{33} = \chi + \frac{1}{2}\left(\chi^2 + \varphi^2\right), \qquad \kappa_{22} = \kappa_2(1+\varepsilon_2) + \theta_2\kappa_{23} \tag{2b}$$

where

$$\varepsilon_2 = \frac{1}{A_2}\frac{\partial v}{\partial\varphi} - \frac{w}{R_2}, \qquad \theta_2 = \frac{1}{A_2}\frac{\partial w}{\partial\varphi} + \frac{v}{R_2} \tag{3a}$$

$$\kappa_2 = \frac{1}{A_2}\frac{\partial\psi}{\partial\varphi} - \frac{\chi}{R_2} + \frac{\varepsilon_2}{R_2}, \qquad \kappa_{23} = \frac{1}{A_2}\frac{\partial\chi}{\partial\varphi} + \frac{\psi}{R_2} + \frac{\theta_2}{R_2} \tag{3b}$$

Traditionally, the strain ε_{33} and the variability of the displacement w along the membrane are neglected in the theory of shells. This does not lead to significant errors for basic estimates of the membrane state and for deflections not greater than the thickness of the shell. In studies of the stability and initial postbuckling behavior of cylindrical shells [18,19] it was shown that if $\varepsilon_{33} = 0$, then the function χ cannot be neglected. If we take $\varepsilon_{33} = 0$, the variability of w is determined by the value of

$$\chi = -1 + \sqrt{1-\psi^2} \tag{4}$$

Substituting Eq. (4) in Eq. (2) we obtain

$$\varepsilon_{23} = \psi(1+\varepsilon_2) + \theta_2\sqrt{1-\psi^2} \tag{5a}$$

$$\kappa_{22} = \frac{1}{A_2}\frac{\partial\psi}{\partial\varphi}\left(1 + \varepsilon_2 - \frac{\theta_2\psi}{\sqrt{1-\psi^2}}\right) + \frac{1}{R_2}\left(\varepsilon_2 + 1 - \sqrt{1-\psi^2}\right)(1+\varepsilon_2) + \frac{\theta_2}{R_2}(\psi + \theta_2) \tag{5b}$$

This is an exact analytical expression for the deformation and curvature of the cylinder according to the Timoshenko hypotheses.

The relations of elasticity for an arbitrary shell structure with respect to the shell thickness can be written as [19]

$$T_{22} = C_{22}\varepsilon_{22} + B_{22}\kappa_{22}, \quad M_{22} = B_{22}\varepsilon_{22} + D_{22}\kappa_{22} \tag{6a}$$

$$T_{23} = C_{44}\varepsilon_{23} \tag{6b}$$

where

$$C_{ij} = C_{ij}^{(0)} - 2HB_{ij}^{(0)}, \quad B_{ij} = B_{ij}^{(0)} - 2HD_{ij}^{(0)}, \quad H = \frac{1}{2R_2}, \quad k = 0 \tag{6c}$$

Here, $C_{ij}^{(0)}$ represents the tension-compression stiffness, $B_{ij}^{(0)}$ denotes the coefficient of mutual influence, and $D_{ij}^{(0)}$ denotes the bending stiffness of the reduced surface [20]. Using the principle of possible displacements, the nonlinear equilibrium equations can be obtained in the form

$$\int_0^{\varphi_N}(T_{22}\delta\varepsilon_{22} + M_{22}\delta\kappa_{22} + T_{23}\delta\varepsilon_{23})A_2\mathrm{d}\varphi - q\int_0^{\varphi_N}[(1+\varepsilon_2)\delta w - \theta_2\delta v]A_2\mathrm{d}\varphi = 0 \tag{7}$$

where q is the pressure intensity acting on the shell. Here the pressure is directed towards the center of curvature for both the original and the deformed surface. Substituting Eqs. (2) and (3) into Eq. (7), we can express the principle of possible displacements as

$$\left(T_{22}^*\delta v + T_{23}^*\delta w + M_{22}^*\delta\psi\right)\Big|_0^{\varphi_N} - \int_0^{\varphi_N}\left[\left(\frac{\partial T_{22}^*}{\partial\varphi} - \frac{A_2}{R_2}T_{23}^*\right)\delta v + \left(\frac{\partial T_{22}^*}{\delta\varphi} - \frac{A_2}{R_2}T_{23}^*\right)\delta w + \left(\frac{\partial M_{22}^*}{\partial\varphi} - \frac{A_2}{R_2}\bar{T}_{23}\right)\delta\psi\right]\mathrm{d}\varphi - q\int_0^{\varphi_N}[(1+\varepsilon_2)\delta w - \theta_2\delta v]A_2\mathrm{d}\varphi = 0 \tag{8}$$

where

$$T_{22}^* = T_{22}(1+\varepsilon_2) + T_{23}\psi + M_{22}x_{11} \tag{9a}$$

$$T_{23}^* = T_{23}\sqrt{1-\psi^2} + T_{22}\theta_2 + M_{22}x_{12} \tag{9b}$$

$$M_{22}^* = M_{22}x_{13} \tag{9c}$$

$$\bar{T}_{23} = T_{23}x_{13} + M_{22}x_{14} \tag{9d}$$

and

$$x_{11} = \frac{1}{R_2}\left(2 + 2\varepsilon_2 - \sqrt{1-\psi^2}\right) + k_2 \tag{10a}$$

$$x_{12} = \frac{1}{R_2}(2\theta_2 + \psi) - \frac{k_2\psi}{\sqrt{1-\psi^2}} \tag{10b}$$

$$x_{13} = 1 + \varepsilon_2 - \frac{\theta_2\psi}{\sqrt{1-\psi^2}} \tag{10c}$$

$$x_{14} = \frac{1}{R_2}\left(\theta_2 + \frac{(1+\varepsilon_2)\psi}{\sqrt{1-\psi^2}}\right) - \frac{k_2\theta_2}{\left(1-\psi^2\right)^{3/2}} \tag{10d}$$

From Eq. (8), we obtain three equilibrium equations

$$\frac{\partial T_{22}^*}{\partial \varphi} - \frac{A_2}{R_2}T_{23}^* - A_2 q\theta_2 = 0 \tag{11a}$$

$$\frac{\partial T_{23}^*}{\partial \varphi} + \frac{A_2}{R_2}T_{23}^* + A_2 q(1+\varepsilon_2) = 0 \tag{11b}$$

$$\frac{\partial M_{22}^*}{\partial \varphi} - A_2\bar{T}_{23} = 0 \tag{11c}$$

as well as boundary conditions with respect to the efforts T_{22}^*, T_{23}^*, and moment M_{22}^*. Using the relations in Eqs. (2), (5), and the system in Eq. (11) with associated boundary conditions at $\varphi = 0$ and $\varphi = \varphi_N$, it is possible to study the stress-strain state of closed and open cylindrical shells of infinite length.

In the case of small angles of rotation, the equations can be considerably simplified by neglecting higher order terms in the Taylor series expansion of the transverse shear ε_{23} and the curvature κ_{22}. Restricting ourselves to the linear expansions for ε_{23} and κ_{22} we have

$$T_{22}^* = T_{22} + \frac{1}{R_2}M_{22}, \quad T_{23}^* = T_{23} + \left(T_{22} + \frac{1}{R_2}M_{22}\right)\theta_2 \quad M_{22}^* = M_{22} \tag{12}$$

3 The Governing System of Equations

To solve the system in Eq. (11), we first introduce the nondimensional parameters

$$\bar{v} = \frac{vR_0}{t^2}, \quad \bar{w} = \frac{w}{t}, \quad \bar{\psi} = \frac{R_0}{t}\psi, \quad h = \frac{t}{R_0} \tag{13a}$$

$$t_{22} = \frac{T_{22}}{C_{22}}\frac{R_0^2}{t^2}, \quad t_{23} = \frac{T_{23}}{C_{22}}\frac{R_0^2}{t^2}, \quad \bar{t}_{23} = \frac{\bar{T}_{23}}{C_{22}}\frac{R_0^2}{t^2} \tag{13b}$$

$$\bar{\varepsilon}_{22} = \varepsilon_{22}\frac{R_0^2}{t^2}, \quad \bar{\varepsilon}_{23} = \varepsilon_{23}\frac{R_0}{t}, \quad \bar{\kappa}_{22} = \frac{R_0^2}{t}\kappa_{22} \tag{13c}$$

$$\alpha_{44} = \frac{D_{22}}{C_{22}t^2}, \quad \alpha_{77} = \frac{C_{44}R_0}{C_{22}t}, \quad m_{22} = \frac{M_{22}R_0^2}{C_{22}t^3} \tag{13d}$$

$$m_q = \frac{qR_0^3}{C_{22}t^2} \tag{13e}$$

In this case, the expressions for deformations in terms of the displacements take the form

$$\bar{\varepsilon}_{22} = \bar{\varepsilon}_2 + \frac{1}{2}\left(h^2\bar{\varepsilon}_2^2 + \bar{\theta}_2^2\right) \tag{14a}$$

$$\bar{\varepsilon}_{23} = \left(1 + h^2\bar{\varepsilon}_2\right)\bar{\psi} + \bar{\theta}_2\sqrt{1-h^2\bar{\psi}^2} \tag{14b}$$

$$\bar{\kappa}_{22} = k_2\left(1 + h^2\bar{\varepsilon}_2 - h^2\frac{\bar{\theta}_2\bar{\psi}}{\sqrt{1-h^2\bar{\psi}^2}}\right) + \frac{h_2}{h^2}\left(1+h^2\bar{\varepsilon}_2\right)\left(1 + h^2\bar{\varepsilon}_2 - \sqrt{1-h^2\bar{\psi}^2}\right) \tag{14c}$$

$$\bar{\varepsilon}_2 = \frac{1}{a_2}\frac{\partial \bar{v}}{\partial \varphi} - \frac{\rho_2}{h}\bar{w}, \quad \bar{\theta}_2 = \frac{1}{a_2}\frac{\partial \bar{w}}{\partial \varphi} + h_2\bar{v} \tag{14d}$$

where $a_2 = A_2/R_0, \rho_2 = R_0/R_2$, and $h_2 = t/R_2$. The relations of elasticity and the equilibrium equations can now be represented as

$$t_{22} = \bar{\varepsilon}_{22} + \alpha_{23}\bar{\kappa}_{22}, \quad \bar{t}_{23} = \alpha_{77}\bar{\varepsilon}_{23}, \quad m_{22} = \alpha_{23}\bar{\varepsilon}_{22} + \alpha_{44} + \bar{\kappa}_{22} \tag{15}$$

and Eq. (11) is of the form

$$\frac{\partial t_{22}^*}{\partial \varphi} - a_2\rho_2\bar{t}_{23}^* - ha_2 m_q\bar{\theta}_2 = 0 \tag{16a}$$

$$\frac{\partial \bar{t}_{23}^*}{\partial \varphi} + a_2\rho_2 t_{22}^* + a_2 m_q\left(1 + h^2\bar{\varepsilon}_{22}\right) = 0 \tag{16b}$$

$$\frac{\partial m_{22}^*}{\partial \varphi} - \bar{t}_{23} = 0 \tag{16c}$$

We now omit bar notation above dimensionless parameters and introduce the following y-notation for our desired functions:

$$y_1 = t_{22}^*, \quad y_2 = t_{23}^*, \quad y_3 = m_{22}^*, \quad y_4 = v, \quad y_5 = w, \quad y_6 = \psi \tag{17}$$

It is then possible to express the functions in Eq. (17) and the load m_q in parametric form

$$y_i = y_i(\lambda), \quad m_q = m_q(\lambda), \quad \text{for} \quad i = 1, \ldots, 6 \tag{18}$$

where the parameter λ satisfies the following condition [15]

$$\sum_{i=1}^{6}\left(\frac{\partial y_i}{\partial \lambda}\right)^2 + \left(\frac{\partial m_q}{\partial \lambda}\right) = 1 \tag{19}$$

In this form, the parameter λ determines the length of a curve described in 7-dimensional space by Eq. (18). If we differentiate Eqs. (14) and (15) with respect to λ, we find

$$a_{ij}\dot{x}_j = \omega_i\dot{y}_i + b_i\dot{y}_6, \qquad i,j = 1, \ldots, 6 \tag{20}$$

where $x_1 = t_{22}, x_2 = t_{23}, x_3 = m_{22}, x_4 = \varepsilon_2, x_5 = \theta_2, x_6 = \kappa_2$ and derivatives with respect to λ are represented by dot notation. In Eq. (20), the summation is made over j for a given i. The coefficients a_{ij} are given by the expressions

$$a_{11} = 1 + h^2\varepsilon_2, \quad a_{12} = h\psi, \quad a_{13} = x_{11} \tag{21a}$$

$$a_{14} = h^2(t_{22} + h_2 m_{22}), \quad a_{16} = h^2 m_{22} \tag{21b}$$

$$a_{21} = h\theta_2, \quad a_{22} = \sqrt{1-h^2\psi^2}, \quad a_{23} = x_{12} \tag{21c}$$

$$a_{25} = h^2(t_{22} + 2h_2 m_{22}), \quad a_{26} = -\frac{h^3\psi m_{22}}{\sqrt{1-h^2\psi^2}} \tag{21d}$$

$$a_{33} = x_{13}, \quad a_{34} = h^2 m_{22}, \quad a_{35} = -\frac{h^2\psi m_{22}}{\sqrt{1-h^2\psi^2}} \tag{21e}$$

$$a_{41} = \frac{\alpha_{44}}{\Delta}, \quad a_{44} = -\left(1 + h^2\varepsilon_2\right), \quad a_{45} = -\theta_2 \tag{21f}$$

$$a_{52} = \frac{1}{\alpha_{77}}, \quad a_{53} = \frac{\alpha_{22}}{\Delta}, \quad a_{54} = -h^2\psi \tag{21g}$$

$$a_{55} = -\sqrt{1-h^2\psi^2}, \quad a_{63} = -\frac{\alpha_{23}}{\Delta} \tag{21h}$$

$$a_{64} = -x_{11}, \quad a_{65} = -x_{12}, \quad a_{66} = x_{13} \tag{21i}$$

and the coefficients for b_i and ω_i are given by

$$b_1 = -h\left(h\frac{h_2 m_{22}}{\sqrt{1-h^2\psi^2}} + t_{23}\right) \tag{22a}$$

$$b_2 = h^2\frac{\psi t_{23}}{\sqrt{1-h^2\psi^2}} - h\left(h_2 - \frac{h^2\kappa_2}{\left(1-h^2\psi^2\right)^{3/2}}\right)m_{22} \tag{22b}$$

$$b_3 = \frac{h^2 m_{22}\theta_2}{\left(1-h^2\psi^2\right)^{3/2}} \tag{22c}$$

$$b_4 = 0, \quad b_5 = x_{13}, \quad b_6 = x_{14} \tag{22d}$$

$$\omega_1 = \omega_2 = \omega_3 = 1, \quad \omega_4 = \omega_5 = \omega_6 = 0 \tag{22e}$$

Here

$$x_{11} = h_2\left(2 + 2h^2\varepsilon_2 - \sqrt{1-h^2\psi^2}\right) + h^2 k_2$$

$$x_{12} = h\left[h_2(2\theta_2+\psi) - h^2\frac{k_2\psi}{\sqrt{1-h^2\psi^2}}\right]$$

$$x_{13} = 1 + h^2\varepsilon_2 - h^2\frac{\theta_2\psi}{\sqrt{1-h^2\psi^2}}$$

$$x_{14} = h_2\left(\theta_2\frac{(1+h^2\varepsilon_2)\psi}{\sqrt{1-h^2\psi^2}}\right) - h^2\frac{k_2\theta}{\left(1-h^2\psi^2\right)^{3/2}}$$

with the remaining a_{ij} coefficients given by $a_{ij}=0$ and $\Delta = \det[a_{ij}]$. The matrix $[a_{ij}]$ is nonsingular, and therefore the solution of the system in Eq. (20) can be represented as

$$\dot{x}_i = b_{ij}(\omega_j\dot{y}_j) + b_{ij}b_j\dot{x}_6, \qquad i,j = 1,\dots,6 \tag{23}$$

where the coefficient matrix for b_i is given by $[b_{ij}] = [a_{ij}]^{-1}$. In Eq. (23), the summation is made over j for a given i. The first three equations of the system in Eq. (23) are expressions of the derivatives $\dot{t}_{22}, \dot{t}_{23}, \dot{m}_{22}$, and these, in turn have been expressed in terms of the derivatives of our desired functions. The following three equations are analogous expressions involving the derivatives $\dot{\varepsilon}_2, \dot{\theta}_2, \dot{\kappa}_2$. If we take into account that

$$\dot{\varepsilon}_2 = \frac{1}{a_2}\frac{\partial\dot{v}}{\partial\varphi} - \frac{\rho_2}{h}\dot{w}, \quad \dot{\theta}_2 = \frac{1}{a_2}\frac{\partial\dot{w}}{\partial\varphi} + h_2\dot{v}, \quad \dot{\kappa}_2 = \frac{1}{a_2}\frac{\partial\dot{\psi}}{\partial\varphi} \tag{24}$$

then the last three equations of Eq. (23) can be represented as follows

$$\frac{1}{a_2}\frac{\partial y_4}{\partial\varphi} = \frac{\rho_2}{h}\dot{y}_5 + b_{4j}(\omega_j\dot{y}_j) + b_{4j}b_j\dot{x}_6 \tag{25a}$$

$$\frac{1}{a_2}\frac{\partial y_5}{\partial\varphi} = -h_2\dot{y}_4 + b_{3j}(\omega_j\dot{y}_j) + b_{5j}b_j\dot{x}_6 \tag{25b}$$

$$\frac{1}{a_2}\frac{\partial y_6}{\partial\varphi} = b_{6j}(\omega_j\dot{y}_j) + b_{6j}b_j\dot{x}_6 \tag{25c}$$

Here the summation is also made over j. Differentiating the equilibrium equation in Eq. (16) with respect to λ, we find that

$$\frac{1}{a_2}\frac{\partial y_1}{\partial\varphi} = \rho_2\dot{y}_2 + h\dot{m}_q\theta_2 + hm_q\dot{\theta}_2 \tag{26a}$$

$$\frac{1}{a_2}\frac{\partial y_2}{\partial\varphi} = -\rho_2\dot{y}_1 - \dot{m}_q\left(1+h^2\varepsilon_2\right) - m_q h^2\dot{\varepsilon}_2 \tag{26b}$$

$$\frac{1}{a_2}\frac{\partial y_3}{\partial\varphi} = \frac{1}{h}\dot{\bar{t}}_{23} \tag{26c}$$

Incidentally, differentiating Eq. (20) with respect to parameter λ allows us to obtain an expression for $\dot{\bar{t}}_{23}$. This expression will contain derivatives of $\dot{t}_{22}, \dot{t}_{23}, \dot{m}_{22}, \dot{\varepsilon}_2, \dot{\theta}_2$, and $\dot{\kappa}_2$. Substituting the values from Eq. (23) into Eq. (26), that is, $\dot{\bar{t}}_{23}$ and $\dot{\theta}_2$ into the first and $\dot{\varepsilon}_2$ into the second equation (Eq. (26)), we obtain three linear differential equations with respect to λ for our desired functions $\dot{y}_1, \dot{y}_2$, and $\dot{y}_3$. The full system of differential equations consists of six equations (Eqs. (25) and (26)). These desired functions must also satisfy boundary conditions at $\varphi=0$ and $\varphi=\varphi_N$. The first boundary condition is expressed in terms of the functions y_1, y_2, and y_3, whereas the second boundary condition is expressed purely in terms of y_4, y_5, and y_6.

The system of Eqs. (25) and (26) is of identical form to the equations for the increments, using the incremental approach [15]. The procedure for solving problems by the incremental approach with unknown loading rate is presented in Refs. [21,22]. Here we multiply each of these equations by $\Delta\lambda$ and replace differentials of functions by finite increments. However, the increment of the load will be written in the form $\Delta m_q = \dot{m}_q\Delta\lambda$. By setting the increment $\Delta\lambda$ we will find the unknown coefficient $\dot{m}_q$ at each step of the loading. Below we use this developed numerical method for calculating the nonlinear deformation of corrugated cylindrical panels, made of fiber composites. Firstly, we will try to obtain an analytical solution to determine the stability of isotropic cylindrical shells under an external load. An analogous problem was solved in classical form by Ref. [1].

4 Stability of an Open Circular Cylindrical Shell

The system of neutral equilibrium equations can be derived from the nonlinear Eq. (16), and we introduce the superscript$^{(1)}$ to represent this analytical solution. If we assume that the membrane is in a subcritical state, the system can be written in the form [19]

$$\frac{\partial t_{22}^{(1)}}{\partial\varphi} + h\frac{\partial m_{22}^{(1)}}{\partial\varphi} - t_{23}^{(1)} - h^2 m_q\frac{\partial\varepsilon_2^{(1)}}{\partial\varphi} = 0 \tag{27a}$$

$$\frac{\partial t_{23}^{(1)}}{\partial\varphi} + t_{22}^{(1)} + hm_{22}^{(1)} - h^2 m_q\frac{\partial\theta_2^{(1)}}{\partial\varphi} = 0 \tag{27b}$$

$$\frac{\partial m_{22}^{(1)}}{\partial\varphi} - \frac{1}{h}t_{23}^{(1)} = 0 \tag{27c}$$

If we take into account the relationship between the bending moment $m_{22}^{(1)}$ from the first equation above, the shearing force $t_{23}^{(1)}$ from the third equation of the system, and that $t_{22}^{(1)} = \varepsilon_2^{(1)}$, then we obtain

$$\frac{\partial\varepsilon_2^{(1)}}{\partial\varphi} = 0 \tag{28}$$

Since at the end of the arc $w^{(1)}=0$, then $\varepsilon_2^{(1)}=0$, and consequently $t_{22}^{(1)}=0$. The remaining two equations

$$\frac{\partial^2 m_{22}^{(1)}}{\partial\varphi^2} + m_{22}^{(1)} - m_q\left(\frac{\partial^2 w^{(1)}}{\partial\varphi} + w^{(1)}\right) = 0 \tag{29a}$$

$$\frac{\partial^2 m_{22}^{(1)}}{\partial\varphi^2} - \frac{1}{h}t_{23}^{(1)} = 0 \tag{29b}$$

can be written in terms of displacements

$$\alpha_{44}\left(\frac{\partial^3\psi^{(1)}}{\partial\varphi^3}+\frac{\partial\psi^{(1)}}{\partial\varphi}\right)-m_q\left(\frac{\partial^2 w^{(1)}}{\partial\varphi^2}+w^{(1)}\right)=0 \quad (30a)$$

$$\alpha_{44}\frac{\partial^2\psi^{(1)}}{\partial\varphi^2}-\frac{1}{h}\alpha_{77}\left(\psi^{(1)}+\frac{\partial w^{(1)}}{\partial\varphi}+hw^{(1)}\right)=0 \quad (30b)$$

These equations can be further reduced to a single equation with respect to the function $\psi^{(1)}$

$$\frac{\partial^3\psi^{(1)}}{\partial\varphi^3}+\lambda^2\frac{\partial\psi^{(1)}}{\partial\varphi}=0 \quad (31)$$

where

$$\lambda=\alpha_{77}\frac{1+m_q/\alpha_{44}}{\alpha_{77}-hm_q} \quad (32)$$

Assuming that solutions of Eq. (31) are of the form $\psi^{(1)}=De^{k\varphi}$, the general solution can be obtained as

$$\psi^{(1)}=D_0+D_1\sin\theta\varphi+D_2\cos\lambda\varphi \quad (33)$$

At the hinged ends of the shell (when $\varphi=0$ and $\varphi=\varphi_N$), we impose the conditions

$$\left.\frac{\partial\psi^{(1)}}{\partial\varphi}\right|_{\varphi=0}=D_1=0 \quad (34a)$$

$$\left.\frac{\partial\psi^{(1)}}{\partial\varphi}\right|_{\varphi=\varphi_N}=\lambda D_1\cos\theta\varphi_N-\lambda D_2\sin\lambda\varphi_N=0 \quad (34b)$$

Assuming $D_2\neq 0$, this implies that

$$\sin\lambda\varphi_N=0 \quad (35)$$

Using Eq. (30), it can be shown that

$$v^{(1)}=B(1-\cos\lambda\varphi_N),\quad w^{(1)}=C\sin\lambda\varphi_N \quad (36)$$

where B and C are nonzero, since they can be expressed with respect to D_2. For the boundary condition

$$v^{(1)}=0,\quad w^{(1)}=0 \quad (37)$$

when $\varphi=0$ and $\varphi=\varphi_\mathrm{N}$, we require not only Eq. (35) to be satisfied but also that

$$1-\cos\lambda\varphi_N=0 \quad (38)$$

The minimum root for these conditions is given by $\lambda\varphi_N=2\pi$. Taking into account this value of λ we obtain

$$m_q=\alpha_{44}\left(\frac{4\pi^2}{\varphi_N^2}-1\right)p \quad (39a)$$

where

$$p=\frac{1}{1+(4h\pi^2\alpha_{44})/(\varphi_N^2\alpha_{77})} \quad (39b)$$

When $\varphi_N=\pi$, the critical load for the panel coincides with the critical load for a closed shell [23]. The same result occurs in the classical solution [1]. The formula in Eq. (39a) will be used to validate the critical load values obtained numerically.

5 Numerical Evaluation

We consider the stability and postbuckling behavior of corrugated shells, which are of interest for practical purposes. For long pipelines, transverse corrugation cannot be used, as pipelines must have high critical values of external pressure and be sufficiently stable under axial compression and bending. Therefore, we consider cylindrical shells, corrugated only in the longitudinal direction. Based on previous studies [10,11], we consider shells with a cross-section formed by arcs of circles of radius r, simply supported on a circle of radius R_0 at the points of conjugation, with $r<R_0$.

We can describe each corrugated shell in term of two angles: φ_0 for the central circle centered about the origin, and γ_0 for the smaller circle which describes the corrugation (Fig. 1). This figure shows that the angle between the radii of the circles labeled as x equals $(\gamma_0-\varphi_0)/2$, and that for conjugate arcs at point C the angle between the radii O_1C and O_2C is equal to $\gamma_0-\varphi_0$.

This means that the tangent to the guide cylinder at the point C during the transition from the first to the second arc turns stepwise by $\gamma_0-\varphi_0$. Using the Runge–Kutta method outlined in Refs. [21,22] on the system described by Eqs. (25) and (26), the values of unknowns at the $(n+1)$th step can be calculated using data obtained at the previous nth step. In reference to Fig. 2, suppose that the nth step is located at the point C on the first circle, and the $(n+1)$th step is located at the beginning of the second circle on (Fig. 3). In the first case, elements of the displacement vector, directed along the normal w_1 and the tangent vector v_1 to the circle, are centered about O_1. In the second case, the normal and tangent vectors to the circle are centered about the point O_2. From Figs. 2 and 3, the compatibility condition of the displacements at the point C can be written as

$$v_1\cos x-w_1\sin x=v_2\cos x+w_2\sin x \quad (40a)$$

$$v_1\sin x+w_1\cos x=-v_2\sin x+w_2\cos x \quad (40b)$$

Solving this system, we find the displacements v_2 and w_2 that are required at the initial point of the second arc. That is, we obtain

$$v_2=v_1\cos(2x)-w_1\sin(2x) \quad (41a)$$

$$w_2=v_1\sin(2x)+w_1\cos(2x) \quad (41b)$$

Taking into account the equalities in Eqs. (40a) and (40b), the conditions of transition from the nth point to the $(n+1)$th point for the vector $\mathbf{Y}$ of resolving functions (Eq. (17)) takes the form

$$\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\\y_6\end{bmatrix}^{n+1}=\begin{bmatrix}\cos(\gamma_0-\varphi_0) & -\sin(\gamma_0-\varphi_0) & 0 & 0 & 0 & 0\\ \sin(\gamma_0-\varphi_0) & \cos(\gamma_0-\varphi_0) & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos(\gamma_0-\varphi_0) & -\sin(\gamma_0-\varphi_0) & 0\\ 0 & 0 & 0 & \sin(\gamma_0-\varphi_0) & -\cos(\gamma_0-\varphi_0) & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\\y_6\end{bmatrix}^{n} \quad (42)$$

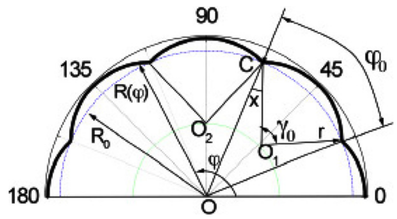

Fig. 1 Cross section of a cylindrical shell corrugated in the longitudinal direction ($N = 5$)

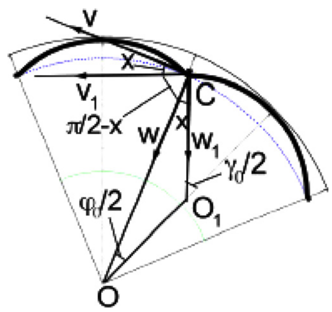

Fig. 2 Arc segment of a cylindrical corrugated shell for the *n*th iteration

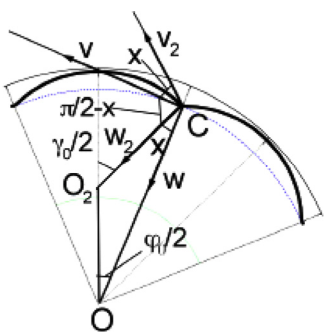

Fig. 3 Arc segment of a cylindrical corrugated shell for the ($n + 1$)th iteration

or in matrix form: $\mathbf{Y}^{n+1} = \mathbf{A}\mathbf{Y}^n$. Note that the use of the presented computational procedure requires that the length of the arc of a single panel should not be less than one-tenth of its thickness. This requirement follows from a known phenomenon of shell theory, as specified in Ref. [24]. The radius $R(\varphi)$ which describes the cross-sectional shape, can be defined as

$$R(\varphi) = R_0 \rho(\varphi) \tag{43}$$

where

$$\rho(\varphi) = w \cos\varphi + \sqrt{d^2 - \omega^2 \sin^2 \varphi} \tag{44a}$$

$$\omega = \cos\frac{\varphi_0}{2} - p\cos\frac{\gamma_0}{2}, \quad d = \frac{r}{R_0} \tag{44b}$$

when $-\varphi_0/2 \le \varphi \le \varphi_0/2$. The Lame parameter A_2 and the radius of curvature R_2 of the curve (Eq. (43)) are found using the formulas

$$A_2 = R_0 a_2, \quad R_2 = R_0 \rho_2 \tag{45}$$

where

$$a_2 = \frac{d\rho(\varphi)}{\sqrt{d^2 - \omega^2 \sin^2 \varphi}}, \quad \rho_2 = \pm d \tag{46}$$

These expressions are valid for convex corrugations. It is interesting to note that noncircular shells of this type have constant curvature, but that the parameter A_2 is a variable. From a mathematical perspective, the discontinuity of the derivative at the point of conjugation is removable. Longitudinally corrugated shells of this type combine the properties inherent for long cylindrical shells with a circular cross-section of radius R_0 and the cylindrical panels of radius r, such that

$$r \sin\frac{\gamma_0}{2} = R_0 \sin\frac{\varphi_0}{2} \tag{47}$$

Since $\varphi_0 \le \gamma_0 \le \pi$ and $0 \le \varphi_0 \le \pi$, then for noncircular shells, $r < R_0$. It follows from Eq. (39*a*) that the critical pressure for panels of radius r is greater than for the shells of radius R_0. Now we consider how this affects the stability of open cylindrical shells consisting of N sections. Note that the solution method outlined here is only applicable to corrugated shells where $N \le 18$ as for $N > 18$ the error in the calculations becomes significant [24].

6 Calculation and Analysis of Results

For given values of R_0 and φ_0, we will alter the angle γ_0 which determines the radius r by means of the relation in Eq. (47). We specify a radius $R_0 = 1$ m, thickness $t = 0.01$ m, $0 \le \varphi \le \pi$, $N = 18$, $\varphi_0 = \pi/N$. The cross section of such a shell for $\gamma_0 = 17\pi/18$ is shown in Fig. 4.

First we consider an isotropic shell with $E = 0.743 \cdot 10^5$ MPa and $\nu = 0.32$. To estimate the critical pressure, we use the parameter $\alpha_c = q_k/q_c$. In Fig. 5, curve 1 illustrates the dependence of α_c against γ_0 for a corrugated isotropic shell, and for $\gamma_0 < 0.55$ and $\gamma_0 > 2.4$ we observe that the parameter $\alpha_c < 1$. This suggests that

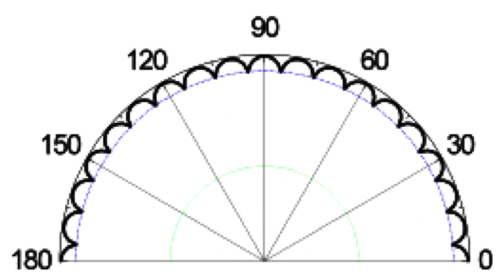

Fig. 4 Cross section of a cylindrical shell for $\gamma_0 = 17\pi/18$, $N = 18$

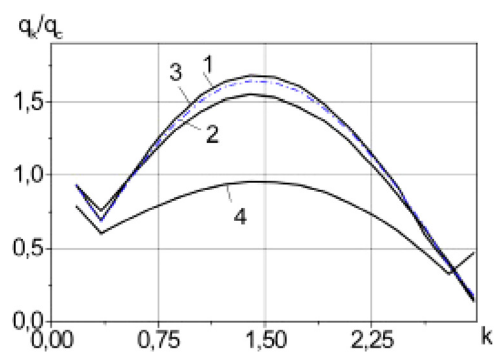

Fig. 5 Relationship between loading and the interior angle γ_0

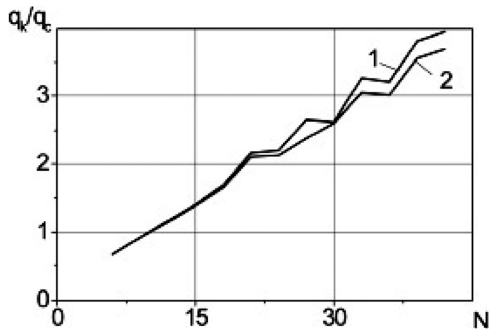

Fig. 6 Relationship between loading and the number of arc segments N

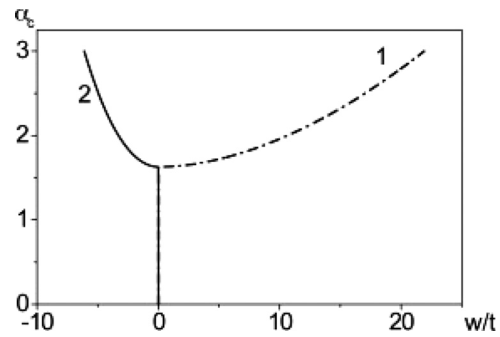

Fig. 8 Relationship between the deflection and load at the points $\varphi = \pi/2$ (curve 1) and $\varphi = 3\pi/4$ (curve 2)

in this range the critical load of corrugated shells is smaller than the critical load for a standard circular shell. However, for $0.55 < \gamma_0 < 2.4$, corrugation increases the buckling load at which there is a loss of stability. Here the maximum value of α_c occurs at γ_0 close to $\pi/2$. Curve 2 in Fig. 5 is obtained for an isotropic shell with $t = 0.05$ m. For this shell, we have $q_c = 2.6$ MPa, and one can see that curve 2 is qualitatively similar to curve 1 (however, it has slightly lower values over this range). Introducing the notation $\alpha_{c,i}$ to represent the ith curve for the parameter α_c, we observe that the maximum value of $\alpha_{c,1} = 1.68$ and $\alpha_{c,2} = 1.55$ differ by just over 8%. Curves 3 and 4 are similar to curves 1 and 2, respectively, but are obtained for thin ($q_c = 0.894$MPa) and thick ($q_c = 12.3$MPa) shells made of composites where $E = 0.3338 \cdot 10^6$ MPa, $G_{23} = 0.1574 \cdot 10^4$ MPa, $\nu_{21} = 0.11$. When $t/R_0 = 0.01$, the values of $\alpha_{c,1}$ and $\alpha_{c,3}$ are close over the entire range of γ_0. Corrugated composite shells for $t/R_0 = 0.05$ have lower critical values of pressure compared to circular shells of radius R_0. When $\gamma_0 = 4\pi/9$ the maximum value of $\alpha_{c,4} = 0.95$. However, thin corrugated shells ($R/t = 100$) made of either isotropic or anisotropic materials, consisting of a cylindrical panel of radius $r < R_0$, can be much more effective than circular shells. Figure 6 shows two curves illustrating the dependence of the parameter α_c on the number of arcs N when $\gamma_0 = 4\pi/9$. The first curve denotes the loading for an isotropic shell, and the second curve shows the loading for an anisotropic shell. These curves are nonsmooth and feature inflection points which are associated with a discrete change in the number of waves where there is a loss of stability.

The difference between these two curves grows as we increase the number of sections; however, they both show the significant effect of corrugation on the stability of long shells. We observe that the critical values of pressure for shells consisting of cylindrical panels are significantly less than the critical pressure for separate hinged sections. In regards to the different forms of stability loss, there is a connection; Fig. 7 shows the undeformed (solid line) and deformed (dashed curve) arc of the cross section by buckling (Fig. 4) when $\gamma_0 = \pi/2$ over the interval $\pi/3 < \varphi < 2\pi/3$. For a single panel, we observe that the buckled shape of the corrugated shell and the asymmetric shape of the buckling (which is connected to a superposition of a local buckling mode on the total buckling mode), are qualitatively similar (see Figs. 4.7–4.9 of Ref. [15]). This fact confirms the correctness of the calculation method used, since the boundary conditions at the ends of each section are asymmetric with respect to the midsection. However, the postbuckling behavior of the individual panels is not the same. Figure 8 shows the two equilibrium curves that describe the relationship between the deflection and load at the points $\varphi = \pi/2$ and $\varphi = 3\pi/4$ for a thin isotropic shell with the same parameters as in Fig. 5. The curves are separated, and at first are almost parallel to the X-axis, gradually curving upwards. This indicates a stable postbuckling behavior for corrugated shells.

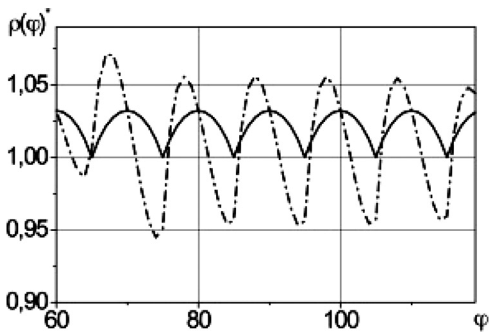

Fig. 7 Deformed (dashed curve) and undeformed buckling (solid curve) over the interval $\pi/3 < \varphi < 2\pi/3$

7 Conclusions

In general, a main property of shells is that transverse loads are balanced by tangential stresses. This property can be used in the construction of designs to obtain interesting deformation features. These designs include cylindrical shells with a corrugated cross section. If cylindrical shells have been corrugated, it is possible to obtain greater critical values of loads, for both axial and surface loading. In this paper, the computational procedures enable us to research the stability and postbuckling behavior of cylindrical shells with this type of longitudinal corrugation. The procedure is based on the equations of nonlinear theory of Timoshenko-type shells with large deflections and angles of rotation, which are derived by the authors. Typical relationships between the parameters of the original shell and cylindrical panels are established, which are useful in designing effective cylindrical shells.

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